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An Examination Of Preservice Teachers' Use Of Learning Trajectories To Guide Instruction

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AN EXAMINATION OF PRESERVICE TEACHERS' USE OF LEARNING TRAJECTORIES TO GUIDE INSTRUCTION

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DEDICATION

This dissertation is dedicated to my loving wife, Michelle Donovan. You have always been a constant foundation of unconditional support and encouragement during the trials and tribulations of both graduate school and life. None of this would have been possible if not for you. I love you. Never forget that I will always be "the greatest fan of your life."

To my children, Bella & Chase. Thank you for finding your way into our lives and for allowing us into yours. I love you both, so much.

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CHAPTER 1: INTRODUCTION

In the National Council of Teachers of Mathematics' *Principles and Standards for School Mathematics* (NCTM, 2000) document, reformers called for mathematics classrooms where "all students should have the opportunity and support necessary to learn significant mathematics with depth and understanding" (p. 50), envisioning classrooms where teachers use students' understandings to guide instruction. This idea was reiterated with the advent of the *Common Core State Standards* (CCSM) (Council of Chief State School Officers & National Governor's Association, 2010). Many of these new standards are "research-based learning progressions detailing what is known today about how students' mathematical knowledge, skill, and understanding develop over time" (p. 4). The question becomes, how do we support teachers in focusing on students' mathematical thinking over time to comply with these new reform movements?

Background – Learning Trajectories

Learning progressions can inform teaching by providing "an empirical basis for choices about when to teach what to whom" … by identifying "key waypoints along the path in which students' knowledge and skills are likely to grow and develop in school subjects" (Daro, Mosher, & Corcoran, 2011, p. 12). In mathematics education, these progressions are known as *learning trajectories* (LTs).

A *Hypothetical Learning Trajectory,* which is a term coined by Simon (1995), describes a path a student might take as they progress toward a learning goal. It is hypothetical in that there is no way for researchers or educators to know exactly how learning will progress for an individual. Rather, there are multiple ways to reach any learning goal, making these trajectories approximations of expected learning. Clements and Sarama (2004) provided three parts necessary for a learning trajectory: "a mathematical goal, a developmental path through which children progress to reach that goal, and a set of activities matched to each of those levels" (p. 3) Daro et al. (2011) said that LTs serve "as a basis for informing teachers about the range of student understanding they are likely to encounter, and the kinds of pedagogical responses that are likely to help students move along" (p. 12).

Learning Trajectories and Preservice Teachers. Mathematics teacher educators face many challenges as they prepare preservice teachers (PSTs) to enter the teaching profession. PSTs enter teacher preparation programs with very little or no experience working with students around mathematical ideas. When given a chance to make sense of students' mathematical thinking, they often use their own reasoning and past learning experiences as a lens and are unable to differentiate the child's thinking from their own. Teacher education programs need to prepare PSTs with an understanding of the informal knowledge children bring with them and how that knowledge can be used as a bridge to more formal mathematical ideas. One model, previously mentioned, that has the potential to guide PSTs through this process is *learning trajectories*.

Problem Statement

Since Simon's (1995) introduction of hypothetical learning trajectories, extensive research has been conducted on students' thinking and how that thinking develops over time. Research findings indicate significant consistency and robustness (Clements & Sarama, 2004; Confrey, Maloney, Nguyen, Wilson, & Mojica, 2008). These findings have helped greatly at the level of curriculum, assessment, and standards development (Corcoran, Mosher, & Rogat, 2009). However, very few studies have addressed the use of LTs as a tool for guiding instruction, calling for a need for further investigation.

The purpose of this research is to investigate how PSTs use, and reflect on their use of, LTs through a one-on-one tutoring project. This research will add to the literature about how LTs are used, and interpreted by, PSTs and also make suggestions about future research to better prepare them for the challenges of creating learning environments that address the thinking of all students.

Purpose and Research Questions

The review of the literature shows that although there is much evidence to support LTs as a guide for understanding student thinking and how it might progress over time, very few studies have addressed the use of LTs during instruction. Even fewer studies have looked at LTs and PSTs. Guided by constructivist learning theory, this study will expand the research findings to elementary PSTs as they use LTs during a one-on-one tutoring project. Part of the reason to study this is to inspire future research that may enhance teacher education programs to better prepare PSTs to identify and respond appropriately to the various levels of student thinking they will encounter in their future classroom. In an attempt to fill gaps in the research literature, this study set out to answer the following research questions:

1) In what ways do PSTs use LTs to assess, plan, and instruct lessons on a geometry topic?
2) In what ways do PSTs reflect on their use of LTs and plan to use LTs to guide their future instruction?

Definitions of Terms

The following terms are particularly important to this study: learning trajectories, students' level of thinking, students' zone of proximal development, and active learning. These terms are defined in this section and will take on these definitions throughout the remainder of this document.

Learning trajectories. Clements and Sarama (2004) theorize learning trajectories as "descriptions of children's thinking and learning in a specific mathematical domain and related, conjectured route through a set of instructional tasks" (p. 83), incorporating learning goals and instruction in students' path of learning. Learning trajectories are comprised of a mathematical goal, domain-specific developmental progressions that children go through, and activities connected with those distinct levels of progression (Clements & Sarama, 2004). This study defines LTs as potential pathways students take as they progress through less sophisticated to more sophisticated understandings of various mathematical content. This definition will be unpacked further in chapter 2. The LT content specific to this study is on geometric shapes.

Students' level of thinking. In this study, this specifically refers to the ordinal number system used in Michael Battista's (2012) *Cognition-Based Assessment & Teaching of Geometric Shapes* (CBA) book. The book contains a diagnostic tool used by the PSTs during the clinical interview with their fifth-grade student. Once the data was collected, the PSTs referred back to the book to identify their students' level of thinking of geometric shapes. PSTs were not prescribed any techniques for calculating the results of the clinical interview to determine their students' level. For example, PSTs were not told that if a student misses a certain number of problems, then they are at a particular level of geometric thinking. Instead, they were given the freedom to compare and contrast the response of their students to the suggested responses from the book, and then determine for themselves their students' level of geometric thinking.

Students' zone of proximal development. According to Vygotsky (1978), the zone of proximal development (ZPD) is "the distance between the actual developmental level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peers" (p. 86). In this study, the students' ZPD included the students' current level of thinking and the level(s) above and beyond that from which the students might work within, successfully, as determined by the PSTs. For example, after determining their student's level of thinking, a PST could conclude that working with concepts two or more levels up might be out of the students' ZPD.

Active learning. In this study, active learning is largely defined as any instructional method that engages students in the learning process, requiring the student to do meaningful learning activities and think about what they are doing. The contrast to this is traditional lecture-style or teaching-by-telling where the student passively receives information from the teacher. Active learning leads to more meaningful learning (Freeman et al., 2014; Henderson, Finkelstein, & Beach, 2010); which has been shown to increase student success and engagement in the classroom (Freeman et al., 2014).

Summary

Teacher educators must develop PSTs' understanding of student thinking if they are to find success in their future classrooms. *Learning trajectories* is a construct that has the potential to engage PSTs in a process of diagnosing student thinking and building upon it, while using research-based ideas of cognitive development. There is still much to be learned about LTs. Teacher educators should enhance PSTs' understanding of mathematical content and pedagogical knowledge; understanding students' mathematical thinking and how it progresses over time is necessary for quality instruction. The purpose of this study is to investigate how PSTs use LTs during instruction. Moreover, this study looks to identify the ways PSTs consider the use of LTs to guide their future instruction.

CHAPTER 2: LITERATURE REVIEW

In this chapter, the theoretical framework that guides this study is presented. Next, there is a review of the literature relevant to this work. The literature review begins with Learning Trajectories (LTs) including their materialization over the years as well as the benefits advocated for teachers and students. After briefly discussing the background of LTs, a review of the research on early childhood education students' and elementary preservice teachers' (PST) levels of geometric thinking is discussed. Also, an examination of the use of tutoring projects during teacher education courses is included. And finally, gaps in the literature on PST's exposure to the aforementioned areas are revealed.

Theoretical Framework

This study takes a constructivist theoretical perspective on teaching and learning. In this section, there is an overview of this perspective and the ideas that are relevant to this study. Of the multiple theoretical viewpoints present in mathematics education, constructivism has dominated the research community over the last several decades (Ernst, 2006). Constructivist posit that the learner actively builds upon their previously formed cognitive structures, or schemas, unique to the individual. The emphasis here is on the belief that students, whether it be children or preservice teachers, come with prior knowledge. The teacher's job is to help students to build upon such knowledge, a theme that continually surfaced throughout this study.

Constructivism. According to constructivist theory of learning, one gains knowledge by acting or otherwise operating on objects and events to discover their properties (Shaffer & Kipp, 2014). Nowadays, it is difficult to find a classroom that does not employ some, if not all, of the ideas based on a constructivist theory of learning. Students of all ages are building and comprehending new knowledge by using manipulatives and other tools in the classroom. Teachers are guiding instead of telling; students are doing instead of watching. Subsequently, the classroom of the 21st century has received a much needed facelift. Although many theorists have developed their own brand of constructivism, there are basic tenets universal with the constructivist theory of learning.

One of the main tenets of constructivist learning theory is that knowledge is constructed, not just simply obtained. Pedagogical approaches that promote "learning by doing" are at the heart of constructivism. The teacher is not seen as the proverbial "vessel that empties information into the minds of the learner", but merely a guide. A guide gives the learner the tools and shows him or her a path toward learning; it is the job of the learner to learn how to learn. Preservice teachers in this study will be "learning by doing" throughout the data collection process. The researcher will play the role of the guide. The PSTs are given the tools (LTs) and, through the tutoring project, are afforded the chance to construct their understanding of the use of LTs during assessment, planning, and instruction.

Jean Piaget is attributed as developing the idea of constructivism. He believed that through the process of accommodation and assimilation, we construct knowledge from our experiences (Shaffer & Kipp, 2014). For Piaget, it is the brain's quest for cognitive equilibrium that drives learning. While other theorists, like Vygotsky, went on to criticize Piaget's stages as being too broad, all maintained that learning happens through our experiences and interactions with our environment. The lens of this study is focused on preservice teachers' leaning, specifically. Consequently, Piaget's stages of cognitive development from birth to adolescence will not be addressed in this literature review. Instead, the focus is on the schemas of preservice teachers as they act on their environment through a one-on-one tutoring project. Piaget addressed how schemas change to achieve balance, or equilibrium, as new knowledge is constructed by the learner.

Piaget viewed knowledge as constructed by the individual through active assimilation (Piaget, 1970). According to this view, schemas are constructed by individuals through experiences connected to their reality, ultimately creating models of reality. When encountering a new situation, the individual searches for internal balance, which Piaget (1975) referred to as equilibration. Equilibration is achieved through assimilation or accommodation. These three processes, assimilation, accommodation, and equilibration, are instrumental to the individual's ability to adapt and learn in their environment.

1. Assimilation: This is the process through which we use our existing mental structures or schemas to take in new information. We need to have prior knowledge to relate to the new information so we can assimilate it. We learn something by connecting new information to something we already know.
2. Accommodation: The process through which our existing mental structures or schemas change as we take in new information. We revise our existing schemas if new information does not fit with them. That is, if we experience something new or different, the mental representations we previously had are changed to acclimate the new experience.
3. Equilibration: This refers to self-regulation, or balancing, that goes on in our minds between assimilation and accommodation. When we start to take in new information, we relate this to what we already know. If this new information somehow differs from existing schemas, the mind is sent into a state of disequilibrium. The mind inherently

> tries to make sense of the new information in order to bring it back to equilibrium. (Piaget, 1975; Shaffer & Kipp, 2014)

For Piaget, "learning is possible only when there is active assimilation" (1964, p. 18). According to Piaget, if new knowledge does not fit into the learner's existing cognitive structures, then a change in these structures is necessary. He called the adjustment of cognitive structures to fit the new knowledge *accommodation*. Assimilation and accommodation describe knowledge in terms of the cognitive building processes. This cognitive building process is a common theme throughout this study.

The learning trajectories themselves, are a cognitive building process since they represent hypothetical paths the brain goes though as it creates more sophisticated understanding about a mathematical concept. The development of these levels of sophistication took years of examining students as they grappled with mathematical content through accommodation and assimilation, making it a tool created through the lens of constructivism. The PSTs in this study experienced similar developmental progressions throughout the twelve-week study as they made sense of the progressions and discovered ways to use them.

For example, consider the fact that not only were the PSTs unfamiliar with the trajectories, but they also had very little experience with teaching prior to this study. If they were only learning about learning trajectories, isolated from this study, that alone would cause a certain level of disequilibrium. Couple that with a tutoring project. It should be safe to assume that more accommodation took place over assimilation since they were trying to blend assessment, planning, instruction, and future use and modify them into their existing schemas, coming to some degree of equilibration.

Piaget is a pioneer in the field of cognitive development. Although other theorists would critique his views for not focusing enough on the impact society has on the learner, this does not mean that he did not consider its implications. Lev Vygotsky felt that this was a concept that needed greater recognition.

Social constructivism. Lev Vygotsky was a Russian psychologist who saw learning as a socio-cultural phenomenon. The learner's interaction with society, in particular, older members with extensive knowledge, is crucial to their cognitive development. Moreover, he felt that:

> Human cognition…is inherently *sociocultural*, affected by the beliefs, values, and tools of intellectual adaptation passed to individuals by their culture. And because these values and tools may vary substantially from culture to culture…neither the course nor the content of intellectual growth was as universal as Piaget's." (Shaffer & Kipp, 2014, p. 231)

According to Vygotsky (1978), new skills are easier to acquire if the learner receives guidance and encouragement from a more knowledgeable other. The PSTs served as the knowledgeable other to their students. Rather being in front of the classroom, they were working side-by-side with the student. This allowed the PSTs the opportunity to not only work with their students in a socio-cultural context, but it also allowed for other Vygotskian approaches to teaching and learning to emerge. The following two central concepts in Vygotsky's theory: zone of proximal development and scaffolding, became evident throughout the study.

- The zone of proximal development: Vygotsky believed that learning takes place when children are working within what he called their *zone of proximal development* (ZPD). According to Vygotsky (1978), the zone of proximal development is "the distance between the actual development level as determined by independent problem solving and the actual level of potential development as determined through problem solving under adult guidance or in collaboration with more practical peers" (p. 86).

- Scaffolding: is the process of providing the adolescent with support during the time they are learning something new. This guidance of the learner from what is known to what is unknown occurs in their zone of proximal development. In contrast to Piaget, Vygotsky believed that instruction precedes development. (Vygotsky, 1978; Bruner, 1986)

The most significant difference occurs in how they approach discovery learning. Piaget encouraged discovery, with little teacher intervention. He wanted more independent, discovery-based activities. Vygotsky, on the other hand, championed guided discovery in the classroom, which involves the teacher asking stimulating questions to students and having them ascertain the answers through exploration (Woolfolk, 2004). In this, the students are engrossed in the discovery process, while simultaneously receiving assistance from a more knowledgeable other.

Since learning trajectories were developed from a constructivist framework, it seems obvious that the PSTs would use them to work within their students' ZPD, giving them better opportunities to scaffold their instruction within the students' zone. In addition, the nature of the one-on-one tutoring environment encouraged a guided discovery approach as the PSTs worked next to their student. This setting does not seem like the right conditions for a PST to take an independent, discovery-based approach with little teacher intervention.

In this study, the researcher is the guide as preservice teachers learn by doing during a one-on-one tutoring project. During the process, the data analyzed looked for times of assimilation, accommodation, and equilibration experienced by the preservice teachers, taking on a constructivist learning theory lens. In addition, this study offered an opportunity to look through a

Vygotskian lens. The preservice teachers acted as the knowledgeable other as they tutored their student. And moreover, through the use of LTs, PSTs were given a potential tool for working within their students' ZPD and for scaffolding their instruction. The focus group interview (discussed later) discussion between the preservice teachers offered further opportunity for socio-cultural learning with each other, but only to a small degree.

Learning Trajectories

The growing body of literature in mathematics education suggests that teachers make pedagogical changes when focusing on student thinking (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Lubienski & Jaberg, 1997). In addition, research suggests that instructional practices that build on students' thinking will foster students' mathematical understanding (Fennema, Carpenter, Franke, Levi, Jacobs, & Empson, 1996; Hiebert & Wearne, 1993). Teachers must begin to understand students' mathematical thinking if they are to create learning opportunities that build upon and support that thinking. LTs were developed to provide a framework for the complex ways students develop knowledge of various mathematical topics. Yet very few studies have explored their potential use as a tool used by PSTs to identify where students fit along such a progression and, in return, how they might guide their decision making process for assessing, planning, and instructing.

Learning trajectories represent children's starting points and the changes that occur during the mathematical activity. In order to build upon students' mathematical knowledge, it is important to consider the gap between the students starting point and the big idea of the mathematics being taught. That is to say, when planning lessons teachers ought to consider how to meet students at their starting points and build them up from there based on what next steps are developmentally appropriate. In recent years, mathematics educators have noted this importance of instructional planning when the goal is to build upon students' current mathematical knowledge (Gravemeijer, 2004). This fits with the Vygotsky's concepts of ZPD and scaffolding.

Background of learning trajectories. Emerging research for many learning scientists is the design of assessments intended to measure student learning along what has been called a *learning trajectory* (LT). LTs are hypothetical paths children pass through in a sequence of thinking. Grounded in research on how children learn and reason in different mathematical concepts, LTs help clarify how student understanding develops over time (Clements & Sarama, 2007; Smith, Wiser, Anderson, & Krajcik, 2006). Specifically, LTs in mathematics denote testable, empirically supported hypotheses about how student understanding develops toward big ideas for learning (Clements & Sarama, 2004; Confrey, Maloney, Nguyen, Mojica, & Myers, 2009; Daro et al., 2011). Confrey, et al. (2009) described LTs as,

> a researcher-conjectured, empirically-supported description of the ordered network of constructs a student encounters through instruction (i.e., activities, tasks, tools, forms of interaction and methods of evaluation), in order to move from informal ideas, through successive refinements of representation, articulation, and reflection, towards increasingly complex concepts over time. (p. 347)

Many different terms have been used to describe the LT construct. For instance, Simon (1995) uses *hypothetical learning trajectories*, Brown and Campione (1996) *developmental corridor*, and Schifter (1998) *big ideas*. While Confrey (2006) designates a *conceptual corridor* and *conceptual trajectory*, the research team would go on to define it as a *learning trajectory* (Confrey et al., 2008). Other terminology such as *learning trajectory* (Clements & Sarama, 2004), *learning performances* (Catley, Lehrer, & Reiser, 2005), and *learning progressions* (Battista, 2004; Corcoran et al., 2009) have been used throughout the literature on the LT construct.

Regardless of the terminology used, a common theme emerges: knowledge progresses from less sophisticated (informal) to more sophisticated (formal) levels of understanding in somewhat predictable ways.

Martin Simon is often cited in the literature as the first to use the term LT in mathematics education. Simon (1995) defined hypothetical learning trajectories as, "The learning goals, the learning activities, and the thinking and learning in which students might engage" (p. 133). The hypothetical learning trajectory offered by Simon (1995) refers to,

> the teacher's prediction as to the path by which learning might proceed. It is hypothetical because the actual learning trajectory is not knowable in advance. It characterizes an expected tendency. Individual students' learning proceeds along idiosyncratic, although often similar, paths. This assumes that an individual's learning has some regularity to it, that the classroom community constrains mathematical activity often in predictable ways, and that many of the students in the same class can benefit from the same mathematical task. (p. 135)

Current uses of LTs add to Simon's seminal work. For example, Clements and Sarama (2004) hypothesized LTs as "descriptions of children's thinking and learning in a specific mathematical domain and related, conjectured route through a set of instructional tasks" (p. 83), incorporating learning goals and instruction in students' path of learning.

According to Clements and Sarama (2004), LTs are comprised of a mathematical goal, domain-specific developmental progressions that children go through, and activities connected with those distinct levels of progression. Furthermore, they describe relationships between mathematical goals and developmental progressions as the "processes involved in the construction of the mathematical goal across several qualitatively distinct structural levels of increasing sophistication, complexity, abstraction, power and generality" (p. 83). Tasks support students' learning at various conceptual levels of understanding within a developmental progression; they identify mental structures that illustrate students' thinking at each level.

Battista (2004) believed that in order for teachers to support students' construction of mathematical learning, they must identify core mathematical concepts, recognize conceptual frameworks for understanding children's thinking, and use appropriate assessment tasks. In his view, LTs provide teachers with information on children's cognitive abilities as well as a structure for assessment. Acknowledging that students have diverse background experiences and mental processes, his model stressed that progression through the terrain may differ, but also that many will pass through the same major landmarks. He calls these well-traveled routes *cognitive itineraries*, which also align with the constructivist framework that drives this study.

The overarching concepts that are essential to future learning and connected to children's thinking are considered the *Big Ideas of Mathematics* (Bowman, Donovan, & Burns, 2001; Clements & Sarama, 2004; Fuson, 2004; Griffin, Case, & Capodilupo, 1995; Tibbals, 2000; Weiss, 2002). In each of these *Big Ideas*, children pass through levels of sophisticated thinking. This concept of progressive development is often associated with hypothetical learning trajectories, descriptions of children's thinking as they move through mathematical domains.

Young children have informal understanding of mathematics that is extensive, complex, and sophisticated (Baroody, 2004; Clements, Swaminathan, Hannibal, & Sarama, 1999; Fuson, 2004; Geary, 1994; Ginsburg, 1977; Kilpatrick, Swafford, & Findell, 2001; Piaget & Inhelder, 1967; Piaget, Inhelder, & Szeminski, 1960; Steffe, 2004). Preschool students, for instance, in free play can be found exploring shapes, patterns, and spatial relations; comparing magnitudes; and counting objects, regardless of socio-economic status or gender (Seo & Ginsburg, 2004). These young children have the ability to engage in high-level mathematical thinking and learning, extending past what is introduced in most programs (Aubrey, 1997; Clements, 1984; Geary, 1994; Griffin & Case, 1997; Klein & Starkey, 2004). Undoubtedly, it is incumbent upon teachers to recognize these high-levels of mathematical thinking, be able to engage in it with the children, and, furthermore, have the awareness, insight, and fortitude to know what is, and how to get them to, the next developmentally appropriate level. LTs have the potential to assist in this process.

This study focused on Battista's (2004) work with *learning progressions* and the various sophistication levels of children's mathematical reasoning. In addition, Battista's (2012) Cognition Based Assessment system was integral to this study. Battista claims this assessment system provides educators with knowledge of mathematical content and how students come to make sense of it. The preservice teachers used this assessment tool to identify student's level of understanding of geometric shapes to guide instructional decisions.

Based on the aforementioned definitions, this study defines LTs as potential pathways students take as they progress through less sophisticated to more sophisticated understandings of various mathematical content. Focused on geometric shapes, the intent of this study was to analyze the ways in which PSTs used LTs during a one-on-one tutoring project and how they reflect on their use, and plan to use them in the future. By viewing this study through a constructivist lens, the researcher was able to identify ways in which PSTs assimilated and/or accommodated new schemas about LTs and their use for teaching and learning, all through active learning, or learning by doing. In addition, the researcher was able to identify ways in which PSTs used, or did not use, Vygotsky's concepts of zone of proximal development and scaffolding to guide their instruction as they played the role of the more knowledgeable other.

Learning trajectories and teaching. Over the past decade, researchers have been analyzing effects LTs have on teaching (Bardsley, 2006; Bargagliotti & Anderson, 2017; Clements, Sarama, Spitler, Lange, & Wolfe, 2011; Edgington, 2012; McCool, 2009; Mojica, 2010; Wickstrom & Langrall, 2018; Wilson, Sztajn, Edgington, & Myers, 2015). In 2006, Bardsley looked to expand the knowledge of early childhood education teacher's understanding of children's mathematical development. The researcher studied the process that 14 pre-kindergarten teachers used while implementing a research-based mathematics curriculum based on LTs. The data revealed that the participants were able to move children through sequences of activities that aligned to help children develop mathematical ideas. Moreover, those who were hoping to gain insight into the children's mathematical development were more likely to use the LT levels in their instruction for grouping students, scaffolding, and to extend their instruction. The study also found that the teacher's views on children's capacity changed as they witnessed them problem solve and learn new, difficult material.

McCool (2009) used a collaborative professional development approach with a single teacher tutoring two students. The professional development focused on LTs for a fifth grade measurement topic to guide instructional decisions. The teacher reported that the LTs enabled her to focus on the students' mathematical thinking and used that information to make instructional decisions.

Mojica (2010), through a design study, identified ways in which preservice elementary teachers used a LT for equipartitioning to build models of student thinking while working with elementary students. The study included 56 participants enrolled in a teacher education program, taking their first elementary mathematics methods course. Findings suggest not only that the PSTs created accurate models of the students' thinking and incorporated those models into their instruction, but also increased the sophistication of their mathematics knowledge for teaching.

Edgington (2012) developed a framework for describing different levels in which teachers used LTs during instruction. Five second grade teachers participated in the multi-case study. The results indicated that teachers used the LT to specify learning goals, pay attention to the process students used while engaging in a task, and to recognize mathematical ideas that arose during instruction.

Wilson, et. al (2015) looked at the ways knowledge of mathematics LTs support teachers with developing student-centered activities. This study was part of a larger design experiment examining teachers learning LTs. They used a 60-hour professional development designed to enhance 22 elementary grade teachers' learning of an equipartitioning LT. Four themes emerged. The LT: assisted the teacher in identifying what they wanted the student to understand; supported the teachers to select cognitively demanding tasks; assisted the teachers in connecting their goals with the curriculum; fostered their ability to create extensions to their tasks.

Bargagliotti and Anderson (2017) studied the effects LTs had when used as a tool for professional development of teachers. Nine secondary mathematics teachers participated in the first iterations in a design experiment of professional development structured around LTs, or what the study called Teacher-Learning Trajectories (TLTs). The study found that LT guided professional development enhanced the teachers' knowledge of statistics and that LTs were used to map the mathematics throughout a curriculum, increasing both the teachers' content and pedagogical knowledge for teaching.

Finally, Wickstrom and Langrall (2018) followed a single teacher who participated in a 10 day professional development on an area measurement LT. The goal of the professional development was to prepare the teacher to use the LT as a formative assessment tool in the classroom. They found that the LT supported the teacher identifying and attending to student thinking, choosing appropriate lesson goals, and modifying tasks, but using the LT to provide differentiated instruction, connect the lesson goals to the goals of the curriculum, and formatively asses her students over time, was not something the teacher chose to do.

Summary. The studies reviewed offer evidence that LTs aid teachers in focusing instruction on children's mathematical thinking. Specifically, Bardsley's (2006) study found that teachers used LTs to not only better understand their students' thinking, but also to create groupings of students and scaffold and extend their instruction; McCool's (2009) participants used their newly acquired knowledge of students' thinking for instructional decisions, while Mojica's (2010) created models of the students' thinking. Another study found that teachers were better prepared to respond to unplanned moments of instruction (Clements, et al., 2011). And Edgington's (2012) study found that LTs helped teachers specify learning goals and recognize student's mathematics thinking during instruction.

More recently, Wilson et. al. (2015) reported finding that LTs helped teachers focus on student thinking and connecting their goals with the curriculum. Bargagliotti and Anderson (2017) found that LTs aided in growing teachers knowledge of the content and that teachers used LTs with curriculum mapping. However, Wickstrom and Langrall (2018), found that while LTs supported the teacher in identifying student thinking, modifying instruction, and choosing appropriate tasks, they were not used for differentiating instruction, formative assessment, or connecting goals with curriculum goals.

The current study builds on the existing knowledge base of teachers' uses of LTs to examine the ways in which PSTs use LTs on geometric shapes as they engage in assessment, lesson planning, and instruction during a one-on-one tutoring project.

Early Childhood Mathematics Education: The Case for Geometry

For early childhood education, geometry is an essential area of mathematics learning (NCTM, 1991, 2006). Broadly speaking, geometric reasoning is not only important in and of itself, but it has been shown to support other mathematical content areas like number and operation concepts and procedures (Arcavi, 2003). Studies even suggest that the capacity to represent magnitude is highly dependent on visual-spatial systems in certain regions of the brain (Geary, 2007; Pinel, Piazza, Le Bihan, & Dehaene, 2004; Zorzi, Priftis, & Umilta, 2002). Regrettably, geometry is often overlooked or minimalized in the school curriculum and by teachers as well (Clements & Sarama, 2009).

An area in geometry often trivialized or overlooked in early childhood and elementary mathematics education is quadrilaterals. Far too many students have experiences with quadrilaterals that lack high cognitive demands (Battista, 2007). In addition, a heavy emphasis on memorization of names and properties of shapes (Burger & Shaughnessy, 1986) has left students with an inconsistent understanding of shapes (Pegg & Davey, 1998) and an inability to identify which properties are relevant to focus their attention. For example, when learning about quadrilaterals, students are often asked questions such as, “Is this a square or a rectangle?” This only requires the student to differentiate certain shapes from others; it does not require higher-order thinking skills needed to understand the attributes of the shapes nor the relationships between them (Clements et al., 1999). A teacher could enhance the question by asking, “Is this a square, rectangle, or both?” … “How do you know?” Many researchers argue that the teaching of geometry should focus not on rote memorization of definitions but rather on tasks that engage the student in the transformation and manipulation of geometric objects, as these tasks provide opportunities for active construction and concrete manipulation, leading to a deeper understanding of geometric concepts (e.g. Battista & Clements, 1988).

Quadrilaterals. There has been much research done in the area of students' understanding of quadrilaterals. The 2009 National Council of Teachers in Mathematics yearbook (*Understanding Geometry for a Changing World, Seventy-first Yearbook)* spent four chapters covering the topic. Evidence suggests three primary reasons for students' difficulties with quadrilaterals. First, students tend to hold stereotypical images of quadrilaterals that are difficult to alter. One example of this is the rectangle with its base oriented horizontally and having a ratio of 2:1 compared to its sides. Students often encounter difficulty when identifying rectangles of varying lengths and orientations (see Figure 1).

Figure 1. Standard rectangle versus non-standard rectangles.

Second, quadrilaterals can be defined by numerous attributes, including side lengths, interior angles, symmetry, diagonal bisectors, congruency, etc. (Casa & Gavin, 2009). Not only are there many attributes to consider, but also the various combinations of those attributes that contribute to the different names and categories of quadrilaterals. This can be another source of confusion for students (Hansen & Pratt, 2005).

Third, students often find it confusing that a square is always (a special case of) a rectangle, but a rectangle is not always a square. Moreover, some curriculums' definitions of quadrilaterals are inclusive, exclusive, or a combination of the two, changing their entire hierarchical structure. For example, an inclusive definition of a trapezoid states that it is a quadrilateral with at least one pair of parallel sides. According to this definition, a parallelogram is always a trapezoid because all parallelograms have at least one pair of parallel sides. Counter to this is the exclusive definition of trapezoids. This definition states that a trapezoid is a quadrilateral with exactly one pair of parallel sides, which excludes parallelograms as a special case of a trapezoid since all parallelograms have two pairs of parallel sides. This inconsistency requires teachers develop students' understanding of hierarchical relationships, with an deep understanding of the role both inclusive and exclusive definitions play. Fujita and Jones (2007) suggests students be able to:

1. Classify shapes in different ways and use different names (i.e., a square is also a rhombus and a parallelogram).
2. Understand transitive relationships between shapes (e.g. if a square is a rectangle, and a rectangle is a parallelogram, then a square is also a parallelogram).
3. Notice asymmetry amongst the relationships between quadrilaterals (i.e., all squares are rectangles, but not all rectangles are squares).

Studies on students' learning of quadrilaterals tend to address the aforementioned issues, focusing on students' progression through the van Hiele levels of geometric reasoning. Before addressing those studies, the following section review gives a brief overview of the van Hiele levels of geometric reasoning.

The van Hiele theory. In the late 1950s, Dutch researchers Dina van Hiele – Geldof and Pierre van Hiele developed a framework for describing and understanding geometric thinking in children and adults, sometimes referred to as the *van Hiele theory*, the *van Hiele model*, or the *van Hiele levels of geometric thought*. The van Hiele model has since become generally accepted as the "industry standard" when studying geometric knowledge. Jaime and Gutierrez (1995) write, "The van Hiele model of [geometric thought] has become a proven descriptor of the progress of students' reasoning in geometry and is a valid framework for the design of teaching sequences in school geometry" (p. 592). Since its inception, our understanding of the van Hiele model has been significantly heightened through the contributions of Usiskin (1982), Mayberry (1981, 1983), Fuys, Geddes, and Tischler (1988), Burger and Shaughnessy (1986), Senk (1989), and Gutierrez, Jaime, and Fortuny (1991), among others.

Van Hiele's (1985) theory uses levels to describe how children develop geometric understanding. There are five van Hiele levels numbered 1 through 5 (Wirzup, 1976); however, the van Hieles initially numbered them from 0 to 4. Hoffer (1981) named the Van Hiele levels, after their inception. The levels are sequential, meaning students must move through prior levels in order to arrive at the next. At Level 1 (*Visualization*), appearance is the dominating factor. Students recognize and name figures based on familiar objects. At Level 2 (*Analysis*), students begin to see the figures as having different properties and are able to consider all shapes within a class. Next, students are able to better understand definitions and categorizations of shapes at Level 3 (*Informal Deduction*). For example, they begin to see the hierarchical relationships between the various shapes, such as knowing that a square is a rhombus. Students rarely reach the Levels 4 and 5 (*Formal Deduction and Rigor*) until high school or even college, if ever. At these levels, students start to reason using axioms and theorems, making conjectures and developing proofs within a mathematical system. Because the study involves PSTs working with elementary students, it will focus on those levels that pertain to students at van Hiele Levels 1 through 3. The next section addresses these levels as described by Battista (2009).

Level 1: *Visualization*, students tend to relate shapes to familiar objects by saying things such as "looks like" or "longer than." For example, they might say that a rectangle looks like a door. Students tend to relate shapes to more familiar ones. They might say that a rectangle is a long square or a parallelogram is a slanted rectangle. Vague language is often used since their vocabulary tends to be underdeveloped. Finally, students tend to think that the orientation of shapes matters.

Level 2: *Analysis*, students “explicitly attend to, conceptualize, and specify shapes by describing their parts and spatial relationships among the parts.” (Battista, 2009, p. 92). By focusing on a class of shapes, they begin to think about what makes a rectangle a rectangle (four sides, four right angles, opposite sides parallel, etc.). They also start to understand the relationships between parts, such as “angles are created by adjacent sides and parallelism.” While they are able to identify a shape as being a part of a class of shapes, they are unable to see hierarchical relationships between classes of figures. For example, they might think that a figure is a square, so it cannot be a rectangle.

Level 3: *Informal Deduction*, hierarchical relationships come to fruition. Students are also able to infer relationships such as, “If a shape has property X, it has property Y.” (Battista, 2009, p. 93). For example, if a shape has four right angles, then opposite sides are congruent. They engage in the reasoning needed for hierarchical classification, knowing that since squares have all right angles and all sides equal and rectangles have all right angles, all square are rectangles, but not all rectangles are squares.

The next section will look at research addressing students’ learning of quadrilaterals and their progression through the van Hiele levels.

Quadrilaterals and the van Hiele levels. Battista (1998) studied 5th graders’ use of Shape Makers, a more basic version of a dynamic software program: The Geometer’s Sketchpad. In The Geometer’s Sketchpad, students choose from a list of shapes and then perform measurement or construction operations on them, changing the shapes by dragging on their vertices. In the study, students were asked to create different types of quadrilaterals by choosing from a list of shapes and dragging the vertices. Battista provided an incident in which students were exploring these various shapes.

Their conversation is as follows:

M.T. - I think maybe you could have made a rectangle.

J.D. - No, because when you change one side, they all change.

E.R. - All the sides [in the Square Maker] are equal. (p. 99)

Battista (2009) stated that M.T.'s comment reflected van Hiele Level 1, because the student displayed visual thinking by believing that a square was distinct from a rectangle. J.D. focused on the movement of the shape, also a level I. The last student, E.R., exhibited level 2 thinking by attending to the components of the shape and abstracting the idea of equal sides. In this study, Battista used the van Hiele levels to illustrate students' reasoning and explained how working in these environments helped students' progression along these levels.

Jones (2000) studied middle school students learning about quadrilaterals in another dynamic geometry software program, Cabri. Students were presented with tasks such as creating *draggable* squares (preserving the shape of the square regardless of how it is dragged) and creating a square and explaining why all squares are rectangles. Jones noticed that students initially gave simple perceptual observations and progressed to more rigorous explanations with precise mathematical language by the end of the teaching unit.

In a more recent study, Yu and Presmeg (2009) asked seventh grade students to use the shape makers in The Geometer's Sketchpad to construct different quadrilaterals. They found that while some students focused on the holistic figure, others attended to the components. For example, one student tried to use the kite maker to make a parallelogram because they looked alike. On the other hand, another student began to recognize symmetry in the kite, saying, "the kite has to be equal on both sides" (Yu & Presmeg, 2009, p. 117), and could not make a parallelogram.

Studies have been done on students' understanding of quadrilaterals and the use of van Hiele levels to describe students' progression in their understanding of geometric shapes. These findings relate to LTs in that students enter with different cognitive levels and move to more sophisticated ideas. Learning trajectories follow the same developmental path as described by the van Hiele levels, but in greater detail. The van Hiele levels are overarching developmental stages, while the LTs are incremental steps. An analogy to this would be a 5-story building. Each floor of the building is a different van Hiele level of geometric thinking. Learning trajectories are the steps of the stairwell that leads to the next floor. The next section will look at early childhood education PSTs' overall low van Hiele levels and its effect on teaching children. This study did not look at PSTs development of van Hiele levels throughout the study. However, potential participants where given the van Hiele test so that varying levels of geometric thinking would be represented in the study. Moreover, the purpose of this next section is to acknowledge preservice teachers' overall low levels of geometric thinking, which could have negatively impacted their ability to focus on pedagogy while working with students.

Preservice teachers' van Hiele levels. Students often struggle with understanding geometry concepts. This holds true for a majority of elementary PSTs (Cunningham & Roberts, 2010; Duatepe, 2000; Kuzniak & Rauscher 2007; Pickreign 2007; Roberts, 1995) who are at low geometric thinking levels (Duatepe 2000; Roberts 1995). Of all mathematics topics, geometry was the one elementary PSTs claimed to have learned the least about and felt they were least prepared to teach (Jones, Mooney, & Harries, 2002).

Many elementary PSTs are at level 1 (*Visualization)* of the van Hiele model of geometric thinking, identifying and classifying shapes only on the basis of their similarity to other objects (‘‘it must be a rectangle, because it looks like a door’’). It is unlikely that these PSTs will reach level 2 (*Analysis*), lacking the ability to recognize and classify shapes by their properties (Clements, 2003; Swafford, Jones, & Thorton, 1997). One study found that 70% of PSTs were below van Hiele level 3 (*Informal Deduction*), at which people understand relationships between classes of shapes. Moreover, almost two-thirds were at the visual level (1) and some were even at the pre-recognition level (0) (Clements & Sarama, 2009). This can have adverse effects on their future students.

Many early childhood PSTs are not prepared to teach geometry. While level 3 (*Informal Deduction*) is not the goal of early childhood education, there are a couple of reasons why PSTs with low levels of geometric thinking should be of concern to teacher preparation programs. First, teachers at the pre-recognition (0) and visual (1) levels lack the ability to assess and teach children at any level. Effective teachers must know considerably more mathematical content of the age and grade they teach (National Mathematics Advisory Panel, 2008). One would be hard-pressed to find a school administrator or parent who would allow a teacher with a first grade reading level to teach first graders how to read. Disappointingly, this practice is common in early childhood mathematics education. This is unacceptable. Moreover, teachers do not wish to negatively impact future learning, but may in fact do so, unintentionally. For example, a student on a “shape hunt” for rectangles could have been instructed to search for shapes with opposite sides the same. They may bring back a square, only to be told they are incorrect because a square has all sides the same. Not only is the teacher missing out on a teachable moment, but a rejection of the student’s choice could cause issues for future teachers who will have to “unteach” what was learned.

A lack of understanding of geometry and geometry education negatively impacts students. For instance, Thomas (1982) found that kindergarten students had a great deal of knowledge about shapes before instruction began. In one scenario, the teacher elicited and verified the children's prior knowledge but was unable to develop new knowledge. Approximately two-thirds of the interactions between the teacher and student had students repeat things they already knew in a repetitious way as in the following example: Teacher: "Could you tell us what type of shape that is?" Children: "A square." Teacher: "Okay. It's a square." (Clements et al., 2011, p. 136) This illustrates findings that children may already possess the information the teacher is presenting. Furthermore, when teachers did elaborate, their statements often had mathematical inaccuracies, such as claiming that all ''diamonds'' (rhombi) are squares, and that two triangles put together always make a square.

Summary. The literature identifies quadrilaterals as a rich and complex area of mathematics education (Usiskin & Griffin, 2008). More importantly, it identifies misunderstandings both teachers and students have involving types of quadrilaterals and their definitions. Many students' experiences with geometry lack high cognitive demand (Battista, 2007) or rely heavily on memorization (Burger & Shaughnessy, 1986). Hence, students tend to have inconsistent understanding of quadrilaterals (Pegg & Davey, 1998) and often attend to irrelevant properties when identifying shapes (Clements & Sarama, 2000; Monaghan, 2000). Studies have also revealed that teachers have difficulties distinguishing an abstract concept from a drawn example of a quadrilateral (Laborde, 1998) and working within the hierarchical classification of quadrilaterals (Fujita, 2012). It is likely that the difficulties and misunderstandings PSTs experienced with quadrilaterals while they were students may negatively impact their ability to teach these concepts to students. A current review of the literature revealed no studies that focused on LTs, PSTs, and geometric shapes, let alone on hierarchies of quadrilaterals.

Preservice Teachers and Tutoring

One goal of teacher education programs is to prepare PSTs to plan lessons. This, however, is not always an easy process. For some PSTs, it opens the doorway to creativity and experimentation, while others find it extremely challenging, which can be attributed to their novice understanding of student thinking. As teachers plan lessons, they must reflect on students' prior knowledge and consider how students are likely to approach and complete tasks (Ball, 1993). This process cannot occur without teachers understanding how students think about mathematics (Ball, 1993; Steffe, 1991). Ball (1993) called this a "bifocal perspective" in which teachers contemplate the mathematics through the eyes of the learner (p. 159). Teachers must consider the mathematical content and the students' thinking processes; an ongoing cycle of hypothesizing, interaction, reflection, and differentiation needed in order to be effective. This cyclical process places PSTs into terrain they have yet to cover.

Since teachers tend to teach according to how they envision learning taking place, Schifter and Riddle (2004) find this to be a good starting point for professional development. By analyzing students' thinking, PSTs focus their attention on student strategies and begin to make conjectures about student thinking that may guide future instructional decisions (Kazemi & Franke, 2004).

According to constructivist theory, we learn through our experiences – constructing our understanding by connecting new experiences to prior knowledge. It is, therefore, easy to see why field experiences are thought of as the most important component in teacher education programs (Brown & Borko, 1992). It is imperative that these experiences are as constructive as possible if teacher education programs want to develop effective mathematics instructors, consistent with the expectations of NCTM's *Professional Teaching Standards* (1991). That said, this study immerses PSTs in a twelve-week tutoring project in which they go through multiple rounds of assessment, planning, and instruction, using LTs as a guide to inform their teaching.

The National Council for the Accreditation of Teacher Education (NCATE) (2002) advocates the use of field experiences as a highly regarded method to prepare PSTs for the rigors of teaching. NCATE defined field experiences as those in which PSTs observe classrooms, actively engage in the instruction and management of students, and/or provide tutoring to a student or group of students. A field experience that exposes PSTs to diverse settings and affords opportunities to apply instructional practices, increases their ability to differentiate for the learning needs of diverse student populations (Blanton & Pugach, 2007).

Tutoring programs have grown substantially over the last 25 years (Fresko & Chen, 1989) and continue to be essential components of teacher preparation programs. Both tutees and tutors benefit from tutoring programs. Tutees benefit from individualized and supplemental instruction above and beyond what they receive from their classroom teacher(s) (Fresko & Chen, 1989). Tutors reported increase in empathy, altruism and self-esteem (Juel, 1996).

PSTs begin to acquire important skills from successful tutoring experiences, including the experience of working one-on-one with a tutee and the chance to implement - often for the first time - instructional strategies learned in teacher education courses. Cohen (1986) found that PSTs experienced higher cognitive gains because organizing materials to teach often aids in the understanding of the materials. The study also found an increase in the knowledge necessary to make meaningful contributions to struggling students.

In 1995, Harwel examined journals kept by PSTs who tutored for 10 weeks. Her findings suggest PSTs improved their ability to reflect on their work, their decision to become a teacher was reinforced, and their perception of the teaching profession became more realistic. Moreover, they experienced a change in their self-concept as a teacher: in the beginning they were mainly concerned with their relationship with the student, moving to a focus on the content of tutoring sessions, and focusing on the specific needs of the child and selecting activities to accommodate.

In the same year, Hill and Topping (1995) studied changes in the skills of PSTs tutoring in England. Various changes were reported: a better understanding of how children learn, improvement in their ability to explain concepts, and an improved ability to apply their knowledge in different situations.

More recently, Van Laarhoven et al. (2006) studied their preparation of both general and special education PSTs to work in inclusive classrooms. The authors suggested that PSTs learned more from their tutoring experience than their coursework. PSTs felt that more fieldwork and less coursework would have been more helpful. It was concluded that PSTs lessened their fears about teaching diverse learners through their positive experiences and also by learning how to instruct diverse learners.

Summary

Prior research has illustrated the importance teachers' current schemas have on their instructional behaviors in the classroom (Holt-Reynolds, 1992; Nespor, 1987; Richardson, 1996). Additionally, evidence suggests that field-based experiences with PSTs can assist in altering their beliefs or at least begin to reflect on alternative perspectives of teaching. Richardson (1996) indicated that PSTs' prior beliefs were open to change when enrolled in teacher preparation programs committed to recognizing those prior beliefs and provided field-based experiences and opportunities for reflection.

This study immersed PSTs in fieldwork in which they assessed, planned, and instructed during a twelve-week tutoring project. This immersion gave PSTs a "learning by doing" experience with LTs which coincides with the constructivist learning theory that guides this study. The goal was to discover the ways PSTs used LTs throughout the tutoring project and, ultimately, how they might use them in their future classroom.

Key Points from the Literature

Several key points from the literature inform this study. First, studies on teachers' uses of LTs indicate their potential benefit to support teachers in focusing on students' mathematical thinking during instruction (Bardsley, 2006; Bargagliotti & Anderson, 2017; Clements & Sarama, 2008; Clements et al., 2011; Edgington, 2012; McCool, 2009; Mojica, 2010; Wickstrom & Langrall, 2018; Wilson, et. al., 2015). Further research can assist in understanding the specific ways in which teachers and PSTs use trajectories for instruction.

Research on classroom instruction stresses the importance of attending to students' mathematical thinking (Carpenter et al., 1989; Fennema et al., 1996; Franke, Carpenter, Levi, & Fennema, 2001) but additional research is needed to understand more accurately how LTs support teachers and PSTs in this practice. Finally, formative assessment has been acknowledged as a powerful tool to increase student learning (Black & Wiliam, 1998), while LTs have been recognized as supporting teachers in assessing student work in more refined ways and using that knowledge to guide next steps in instruction (Edgington, 2012). This study aimed to add to the existing literature on the uses of LTs to support teaching, in particular, how PSTs' use LTs during teaching (assessment, planning, and instruction) and how they might use them in their future classrooms.

The gap in the literature addressed by this study lies within the intersection between LTs, geometric thinking, and teaching, with PSTs being subjected to all three. Figure 2 is an illustration of this intersection, giving a conceptual look at the purpose of this study.

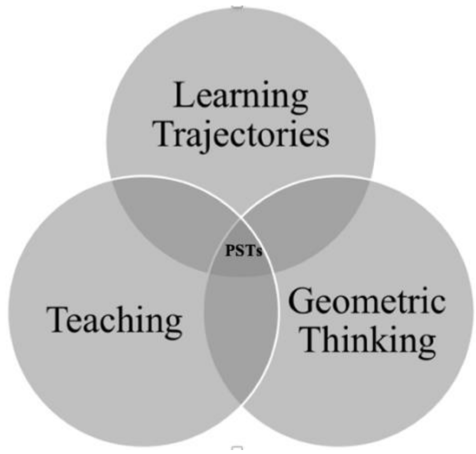


Figure 2. Diagram of the intersection of the major themes of the study.

CHAPTER 3: METHODOLOGY

Introduction

The review of the literature shows that although there is much evidence to support LTs as a guide for understanding student thinking and how it might progress over time, very few studies have addressed the use of LTs as a tool for instruction (Bardsley, 2006; Edgington, 2012; McCool, 2009; Nguyen, 2010). Additionally, there are virtually no studies on the impact LTs have on PSTs during teacher education courses (Mojica, 2010). In keeping with the constructivist learning theory which guides this study, the data collected came from documented scenarios representing times where PSTs were "learning by doing" and given opportunities two assimilate or accommodate new schemas. Moreover, a Vygotskian lens was used to interpret PSTs' work with their students.

The design of this study embodied constructivist learning theory and is described in this chapter. This study will expand research findings to elementary PSTs as they learned about, used, and reflected on their use of LTs during a twelve-week tutoring project. One reason to study this was to consider ways to prepare PSTs to identify and respond appropriately to the myriad of student knowledge they will encounter in their future instructional practice. This study addressed the following questions:

1) In what ways do PSTs use LTs to assess, plan, and instruct lessons on a geometry topic?
2) In what ways do PSTs reflect on their use of LTs and plan to use LTs to guide their future instruction?

The purpose of this chapter is to discuss the methods used to administer this study. A qualitative collective case study approach (also known as a multiple or multi case study) that mirrored Stake's (1995) techniques for research design, data collection, and analysis, was used.

The primary data collection tools were reflective journals, lesson plans, observations, and interviews. All of these were analyzed and sorted into themes and patterns. The results of the study describes preservice teachers' thoughts on instruction while participating in a twelve-week tutoring project focused on using LTs to assess, plan, and instruct students. The next section clarifies the reasons for choosing a collective case study design.

Purpose of the Study

The previous chapter revealed gaps in the literature on the potential use of LTs to guide teacher instruction; there is an even greater gap in the literature on their use with PSTs. Most studies do not include the views of PSTs, with exception to the aforementioned ones. The previous research conducted has not included PSTs participating in a tutoring project using LTs as a tool to guide their process. Additionally, there is limited research on the impact LTs have on PSTs and the way they might be used in their future classrooms. This study intended to add to the current literature.

Research Design

This study used a qualitative research design methodology. The focus of this research could not be put into quantifiable numbers or statistics since there were no known quantitative instruments to measure this study. In addition, the purpose of this study was not to measure PSTs' reaction to the phenomenon, but to make sense of them; the purpose of this study was to identify themes and patterns that are unknown. Therefore, this study required detailed understanding of the cognitive processes which necessitates multiple sources of data, an important characteristic of qualitative research (Stake, 1995; Yin, 2009). This study included multiple sources of data that could lead to contradictions. To address this, Stake's (1995) concept of triangulation was applied to the data collected, which involved comparing and cross checking the data from the multiple sources.

Another reason for choosing a qualitative methodology was that the PSTs' experience would be a complex experience and could not be separated from the context in which the phenomenon occurred. To achieve this the researcher studied the phenomenon in its natural setting (Stake, 1995; Yin, 2003, 2009).

Creswell (1998) recognizes five traditional methodological approaches to qualitative inquiry (i.e., biography, phenomenology, grounded theory, ethnography, and case study) and all were taken into consideration for this study. After a careful review, two of the five approaches were deemed most useful for answering the proposed research questions: phenomenology and case study. Phenomenology attempts to describe or understand the "lived experiences" of the participant, which is not the purpose of this study. Rather, this study looks to develop an in-depth description and analysis of different cases in order to understand PSTs' reactions to, and cognitive processing of, the phenomenon. This level of understanding requires in-depth data collection from multiple sources to paint a holistic picture of the phenomenon. (Stake, 1995; Yin, 2009). Suffice it to say, the characteristics identified lead this study to follow case study design.

The research questions developed for this study asked, *"what"* questions, which are indicative of case study methodology (Stake, 1995; Yin, 2009). The goal of the research was to understand how PSTs interpret their experience. According to Yin (2009), case study methodology "investigates a contemporary phenomenon within its real-life context" (p. 18). Thus, the selection of a case study methodology was supported by the fact that the focus of the research was a contemporary event and would take place in a real-life context.

Once case study methodology was chosen for this study, a further choice was necessary to make. The research literature identifies two major case study models from which to choose: Robert

Stake's or Robert Yin's models. Yin (2003) suggested that a case study is used "to contribute to our knowledge of individual, group, organizational, social, political, and related phenomena" (p. 1). Yin (2003) also asserts that case studies are best suited for a phenomenon like projects or programs. In particular, Yin (2009) states that a case study "tries to illuminate a decision or set of decisions" of the program or project. (p. 17). On the contrary, Stake's (1995) model is more conducive to developing an understanding of a wider range of phenomena. Additionally, Stake (1995) suggested "research questions typically orient to cases or phenomena, seeking patterns of unanticipated as well as expected relationships" (p. 41). While both models seek to understand a specific phenomenon, Stake's method provided an opportunity to categorize patterns among the data, both expected and unexpected. The purpose of this study was to understand the experience of PSTs, not a program or a project; it also aimed at finding expected and unexpected patterns in the data. Stake's case study model was deemed most appropriate to answering the proposed research questions. Not only does Stake (1995) provide detailed instructions on research design, but also in data collection and analysis. This kept the research on track and provided strategies for using a case study approach.

Creswell (2003) noted that a case study is suitable when the study has clearly identifiable cases with boundaries and is intended to provide an in-depth understanding of the cases. This study identified the cases (preservice teachers) and defines the boundaries (participants will be enrolled in the mathematics methods course).

Stake identified three types of case studies. The first two types of case studies are intrinsic and instrumental. The third type, collective, is described in the next paragraph. Stake (1995) proclaims that an intrinsic case study is used when there is "a need for general understanding" (p. 3). He added that the researcher is not interested in a general problem; rather the researcher has an intrinsic need to examine the case (Stake, 1995). Instrumental case studies, on the other hand, identify a general problem. Since this research is not attempting to understand the case for an intrinsic reason, but how each case informs the general problem, this study was an instrumental case study design.

Finally, because the goal is to understand something more than a single case, as defined above, and no two cases of PSTs using LTs during a tutoring project are exactly the same, an examination of multiple cases further enhanced this study. Studying more than a single case made this research a collective case study (Stake, 1995), also referred to as a multiple case study (Yin, 2003). In a collective case study, Stake (1995) states that the research must have multiple cases, similar around a phenomenon, and focused on the research question. Stake (1995) added that collective case studies are typically instrumental case studies that are extended to several cases. Therefore, the methodology for this study was a collective case study based on Stake's (1995) third model.

The role of the researcher. Qualitative research is the study of a social phenomenon in its natural setting and from the participants' perspective. Consequently, the relationship between the researcher and the participants must be considered. According to Patton (1990), "the challenge is to combine participation and observation so as to become capable of understanding the program as an insider while describing the program for outsiders" (p. 207). While staying somewhat removed in order to observe and analyze, the researchers must also engage in a certain amount of participation to experience the phenomenon first hand and to gain the confidence of the participants, which was accomplished throughout the study through observations and interviews.

The researcher's second role in this study was that of interpreter (Stake, 2000), collecting and copying artifacts relevant to this study. This followed Stake's (2000) description of the researcher as an interpreter who recognizes, gathers, and substantiates new meanings.

Prior to this study, the researcher worked with the PSTs as their instructor and department testing administrator. One may argue that because of the researcher's prior experience with them, the PSTs might view the researcher as an authority on mathematical knowledge, causing feelings of inadequacy. However, the researcher argues that because of the researcher's experiences with the PSTs, and the relationships established with them, the PSTs were more apt to open up with thoughtful discussions about their experiences. The researcher intended to reduce the PSTs' view of the researcher's presence as evaluative and more as seeking understanding. In this way, the researcher was able see a more accurate picture of the PSTs' experiences throughout the study.

During observations, the researcher's role was to observe the PSTs' instruction, not to co-teach or coach during instruction. During the interviews, the researcher used the interview process to encourage the PSTs to reflect on their instruction and to clarify their thinking regarding the instructional decisions. The researcher also engaged in dialogue with the teachers as they considered evidence of their students learning to choose future learning goals. The researcher's goal was to build a stronger rapport with the PSTs in order to hold open discussions about not only successful teaching moments but also difficulties they experienced during the study.

According to Merriam (1998), it is necessary to consider how the researcher affects what is being observed. Interactions between researcher and PSTs in qualitative research may bring about changes in both parties. These changes are natural, but must be identified and accounted for by the researcher. In this study, the PSTs' uses of LTs could have change over time. Asking questions specific to how the PSTs were monitoring or how they chose students' solutions to share brought certain issues to the PSTs' attention that they may not have considered. Additionally, over time the PSTs became more comfortable with the researcher's presence during observations and interviews, allowing for a more natural learning environment.

Participants. Participants in this study were chosen from a group of K-8 mathematics education minors at a mid-west university during the fall of 2017. The participants had previously taken all eight of the required mathematics education courses. With this requirement, the participants had already worked with children in different capacities over the course of three to four years. These experiences might have given the participants less anxiety about teaching mathematics, making them more confident to focus on student knowledge during the tutoring project.

The three participants, recruited through purposeful sampling (discussed in the next section), will be referred to by the pseudonyms Melissa, Beth, and Nan throughout the study. They met all of the aforementioned requirements for participation. All three were working toward K-8 teaching certification and attempting to obtain their first bachelor's degree. No other demographic information was collected, since it was not pertinent to this study.

Sampling. In collective case study, selecting participants usually begins with cases that are somewhat identified (Stake, 2006). Purposeful sampling - a sample fitting the study - is used in collective cases studies to select a variety of participants and, in turn, creates opportunities for intensive study (Stake, 2006). Collective case studies aim for greater depth of information rather than breadth of information (Stake, 1995; Yin, 2003). Hence, this study used a maximum variation sampling (Patton, 1990) to make sure the cases were as different from each other as possible.

Van Hiele test. In order to employ maximum variation sampling, this study used the van Hiele test to identify PSTs' levels of geometric thinking. The van Hiele Test comes from the Cognitive Development and Achievement in Secondary School Geometry (CDASSG) project (Usiskin, 1982) and was used to determine prospective PSTs' van Hiele levels. Only PSTs scoring at different levels of the test were chosen as participants. As indicated earlier, Stake (1995) argues that collective case studies produce greater depth and breadth of information when maximum variation sampling is used. Therefore, this study used this form of sampling.

The van Hiele test (see Appendix C) has twenty-five multiple-choice items, with five items per level, and was designed to capture the key characteristic of each van Hiele level. Items 1-5 (Level 1), Items 6-10 (Level 2), Items 11-15 (Level 3), Items 16-20 (Level 4) and Items 21-25 (Level 5). For example, level 1 identifies PSTs' thinking related at which figures are judged according to their appearance; level 2 identifies PSTs' behaviors at which figures are identified for their properties; and level 3, inclusive and exclusive definitions of figures are used to understand relationships between shapes (i.e., a square is always are rectangle because it has four right angles). No PSTs' levels were found to be above level 3, therefore levels 4 and 5 are not discussed.

The three PSTs chosen for the study were at different levels of geometric thinking. Melissa, Beth, and Nan were at levels 1, 2, and 3, respectively. This allowed the researcher to see how PSTs with different levels of geometric understanding experience the phenomenon. Hypothetically speaking, PSTs with higher levels of geometric understanding might rely less on the LTs due to their in-depth understanding of the topic, while PSTs with lower levels may find the LTs useful in helping to develop their own understanding of the geometry concept while they grapple with teaching it. However, this distinction was not specifically addressed during data analysis.

The chosen PSTs had the option to accept or decline without penalty during the recruitment process. Before they decided, they were given detailed information about the purpose of the study and what would be expected of them during the study, informed of the fact that they could drop out of the study at any time and that they were allowed to see any data collected on them at any time during or after the study, and given the results at the end of the study. Each PST was given an informed consent form (see Appendix A) explaining the aforementioned information. Melissa, Beth, and Nan signed the form and joined the study.

Clinical interview. An emphasis on questioning in mathematics teaching (e.g., Carpenter, Fennema, Franke, Levi, & Empson, 2000; Mewborn & Huberty, 1999) supports the idea that questioning is the most commonly used instructional tool (Wassermann, 1991). Research in recent years has shown increased interest in the relationship between teacher questioning and children's thinking (Baroody & Ginsburg, 1990; Buschman, 2001; Fennema, Franke, Carpenter, & Carey, 1993).

Clinical interviews of students, which combines questioning and observation, gave the PSTs a chance to probe their students' mathematical thinking. Piaget (1926, 1929) is credited with development of one-on-one clinical interviewing as a technique for investigating the extent of children's knowledge on many topics, one of which is mathematics. More recently, interviews are also being used as a way to identify students' misconceptions and error patterns (Ashlock, 2002). While few studies have addressed the use of clinical interviewing with PSTs, Moyer and Milewicz (2002) argue that "Structured opportunities that engage preservice teachers in learning appropriate questioning strategies in mathematics and that provide direction in analysis and reflection can be valuable experiences in preparing for future classroom situations" (p. 310).

This study not only had PSTs use clinical interviewing as a way to probe students' thinking, but also looked at how they used those results to guide them through planning, instruction, and then eventually with assessment starting the cycle over. PSTs used Michael Battista's (2012) *Cognition-Based Assessment & Teaching of Geometric Shapes* (CBA) as a diagnostic tool for the clinical interview.

Using the diagnostic tool from the book (see Appendix D), PSTs performed a clinical interview with a 5th grade student and assessed the child's knowledge of geometric shapes, with an emphasis on the relationships of quadrilaterals. The clinical interviews lasted no longer than 25 minutes.

Once the PSTs gathered the data, they referred back to the CBA that same evening to respond to journal prompt 2 (see Appendix E) to determine their students' level of understanding of geometric shapes and provide evidence for their decision. The CBA contains examples of student responses for each question, which gave PSTs a chance to compare and contrast the responses they elicited from their students during the interview. That, along with detailed descriptions from the CBA about each level, gave PSTs a tool to determine their student's LT level, or level of sophistication, for geometric shapes.

Data collection. The main purpose of conducting case study research is to attain the descriptions and interpretations of others. This is important because everyone sees a situation differently and has personal experiences and interpretations unique to them (Stake, 1995). In order to identify these experiences and interpretations of the PSTs in this study, multiple data collection tools were used including reflective journals, lesson plans, observations, and interviews. These tools are described below, followed by Figure 3 (see below), which gives details about which tools were used to answer each research question.

Documents give the researcher the ability to examine how each case (each PST is considered a case) behaved in context (Stake, 2006). They are useful data sources in that they are stable and can be reviewed constantly while covering a long span of time, many events, or situations (Yin, 2003). The researcher revisited the data multiple times from multiple perspectives in the analysis phase, which documents allow (Huckin, 1995). In case study research, documents are often collected in order to verify and substantiate findings from other sources. In this case, they were used to examine the thought processes of the PSTs. While documents have been critiqued in case study research since "they are not always accurate and may not be lacking in bias" (Yin, 2003, p. 87), they are not a weakness in this study; the essence of this study is that of a constructivist nature, and with it, a desire to understand multiple realities of individuals in their context. Documents could reveal the underlying biases, beliefs, and assumptions that influence practice of PSTs, making them essential for this study. The documents reviewed in this chapter are reflective journals, lesson plans, observations, and interviews. See Appendix B for the timeline for data collection.

Reflective journals. During the tutoring project, PSTs made five reflective journal entries articulating their experiences (see Appendix E). The first entry was used as baseline data for the perceptions PSTs brought with them to the study. Later, it was used in conjunction with the final reflection paper as a springboard for further discussion about changes in thinking after using LTs during the tutoring project. This offered the researcher a chance to identify different schemas PSTs created through assimilation and accommodation. The final journal entry also contained one last question addressing how the PSTs might use LTs in their future classrooms, to address the second research question for this study.

Reflective journal prompts 2 – 4 gave snapshots of PSTs thinking throughout the study. As mentioned above, prompt 2 focused PSTs interpretations of the data they collected from their students during the clinical interview. Prompt 3 focused on how the results from the clinical interview and LTs influenced their lesson planning and asked the PSTs to discuss the evidence they would be looking for to determine if the lesson was effective. The fourth journal prompt took place after the lesson and gave PSTs a chance to reflect on what they thought was going to happen, what did happen, what they wish they had done differently, what was most effective, if the LTs supported them during instruction, and how they planned to move forward. While the fifth prompt asked the PSTs to reflect on their experience as a whole, the fourth prompt was a vital piece of data as its purpose was to provide insight into what the PSTs were experiencing and how they were making sense of it. The fourth prompt took place during a time of either assimilation and/or accommodation for the PSTs as it was a heightened time of learning by doing.

Lesson plans. PSTs planned lessons of their choosing and submitted them via email no later than the night prior to the lesson. They were not given a specific lesson plan template in an attempt to give them the freedom to make their own planning decisions without any outside influence from the researcher. For example, the researcher did not want to have them fill out a form that asked them what prior knowledge the student has, which many lesson plans address.

The researcher's initial hypothesis was correct, the PSTs did create each lesson plan based on results from either the clinical interview or preceding lesson, juxtaposed with their growing understanding of LTs for geometric shapes. All lessons addressed each student's level of understanding of geometric shapes. This was not a surprise since they were working one-on-one with a student. The lesson plans had another purpose, as well.

The lesson plans offered the researcher a chance to look for adaptations made during instruction. PSTs' lesson plans provided evidence for what they intended to do in their lessons. The researcher was able to compare and contrast what they planned to do and what actually took place. This provided extra talking points during the interviews to determine why they made any changes they did and to verify whether or not LTs had an influence on these decisions during instruction.

Observations. Observations are a meaningful way of collecting data in any setting and are a critical method in qualitative research (Marshall & Rossman, 2006). Observations can be direct or indirect and formal or informal, all of which can allow the researcher to "know personally the activity and the experience of the case" (Stake, 2006, p. 4). They allow the researcher to see the participants' reality in real time (Yin, 2003) and in context. In addition, the researcher may see things that escape the person's awareness or learn things the person might not have been willing to talk about (Patton, 2002). For example, a PST might not share a misconception they had about quadrilaterals while working with their student. Patton (2002) states that impressions and feelings of what was observed are important to understand as well.

In preservice teacher education, observations are a normal practice and PSTs expect them multiple times throughout their experience. In this study, PSTs were observed on multiple occasions. The purpose was to triangulate data from multiple sources over time. Merriam (1998) stated that long-term observations in qualitative research increases validity of the findings. Data can be confirmed or disproved throughout multiple points in time.

Preservice teachers were made aware that they would be observed during the tutoring project and that the researcher would be attempting to make meaning of their actions. The researcher observed lessons from an outsider perspective (meaning he was not involved in the interaction with the tutor or tutee). The researcher could not catch everything going on, so there must be a focus. The focus guides the study design as well as the nature of the research questions (Patton, 2002). The researcher took a narrowly focused approach to the observations - looking at small parts of what was happening - focusing on ways PSTs used LTs during instruction.

The physical environment can impact what happens (i.e., a crowded room). The clinical interview and lessons with the students took place in an office located inside the elementary school.

This office was away from the classrooms and other potential distractions. The room was approximately 8' x 10' and had two desks in it. While it was a little small, it was big enough for the researcher to observe and take notes without the students or PSTs seeing it take place. The advantage of such a room allowed for audio recordings that were clear enough to make accurate transcriptions of the conversations.

Patton (2002) says to focus on the planned activities in a sequential manner to illustrate the inquiry progress over time, initially focusing on what is said and how participants respond. Basic descriptive questions should guide the researcher through the sequence of the observation. For example, "what is being done and said?", "when do things happen?", and, "what variations do the participants do compared to what they predicted in their pre-lesson form?" Also, "how do the participants bring closure to their instruction?" The researcher did look for, and took note of, the end of instruction to check later if it did and did not connect to future instructional goals.

Each observation and interview was treated as self-contained so the research could manage the fieldnotes (Patton, 2002). They were seen "as a mini-case write-up of a discrete incident, interaction, or event" (Patton, 2002, p. 285), allowing the researcher to look across sessions for patterns and themes during the analysis phase.

Fieldnotes. Future recall cannot be trusted, so fieldnotes were taken during each session. The researcher recognized that he could not write down everything during the sessions. A smart pen was used, which recorded both audio of the conversation and took pictures of the writing simultaneously. This allowed for timestamping when certain phenomenon were observed and it also acted as a backup audio recording alongside the digital voice recorder used. In addition, immediately following each session, the researcher designated an hour to add to the fieldnotes for both interviews and observations. On campus, the researcher was able to do this in his office. Off campus, he requested the continual use of the office space at the elementary school, which was granted by the principal.

Fieldnotes should describe what has been observed, containing everything the researcher feels is important (Patton, 2002). The researcher included direct quotes, paraphrasing, and nonverbal communication, as well as his feelings, reactions to the experiences, and reflections about what was observed. Moreover, the fieldnotes contained insights, interpretations, beginning analysis, and a working hypothesis about what was happening, which Patton (2002) recommends.

Interviews. Each PST participated in three rounds of one-on-one interviews with the researcher, followed by a culminating focus group interview. This section starts with a discussion of the importance of individual interviews, followed by focus group interviews.

Qualitative researchers have the opportunity to focus on what they feel is valuable information and a responsibility to draw meaningful conclusions from that information (Stake, 1995). Interviewing is an appropriate method to unearth that information, and is one method of data collection used in this study. This study used Patton's (2002) standardized open-ended interview method. The purpose for choosing this method of interviewing was that Patton (2002) recommends it for inexperienced interviewers. These types of interviews are more like a script, which gave each PST the same questions in the same order, stimuli, and way.

Stake (1995) suggests the qualitative interviewer prepare for interviews with a list of issue-oriented questions and provide a copy to the participant to view the questions prior to the interview, helping them to consider answers which are more in-depth than just a yes or no response. It should be noted that all three PSTs chose to not look over the questions prior to the interview. Each of them explained, to some degree, that they felt their responses would be less authentic if they looked over the questions ahead of time.

As there are two separate and unique research questions for this study, it was important to design open-ended guiding questions for each one (see Appendix F). The interview questions found in Appendix F were used to gain insight into the first research question: In what ways do PSTs' use LTs during assessment, planning, and instruction. These interviews took place immediately following each PSTs' meeting with their students. While many of the questions were predetermined, other probing questions surfaced before and during the interviews. Reflective journals, lesson plans, and observations were analyzed for patterns and themes prior to each interview, allowing the researcher to determine if there were unanticipated events that need further examination.

The final interview questions (see Appendix F) examined PSTs' reflections on their use of LTs and how they might use them in their future classroom. These interviews took place within a week following both the clinical interview and the first lesson implementation. Many of the questions were predetermined, but other questions were added based off of analysis of the PST's written reflections.

All interviews between the researcher and PSTs were conducted in an office on campus. They were digitally recorded and the researcher took notes using the same aforementioned smart pen.

Focus Group Interview. The final data source for this study was a focus group interview. This data collection method varies greatly in its use but usually involves a small group of interviewees who share characteristics relevant to the study's purpose (Marshall & Rossman, 2006). The purpose of the focus groups is to listen and gather information to gain a greater understanding of the participants' perspectives, thoughts, and ideas on the research topics (Krueger & Casey, 2000). The open-ended format of focus groups allow participants to comment, explain, and share experiences freely without a need for consensus (Krueger & Casey, 2000). The PSTs in this study knew each other relatively well after having spent multiple years together in the teacher education program. Their previous relationships and knowledge of one another made for an easy creation of a cooperative and safe environment, which is necessary for a successful focus group interview (Creswell, 2007; Marshall & Rossman, 2006). This nonjudgmental setting in which the experiences were shared allowed a space for self-disclosure that might not happen in traditional, one-on-one interviews (Krueger & Casey, 2000).

The researcher chose a focus group interview for two reasons. First, the focus group interview method assumes that individuals' attitudes and beliefs do not "form in a vacuum" and that listening to others' ideas benefit all involved (Marshall & Rossman, 2006, p. 114), which fits with the purpose of a collective case study, rather than focusing only on individual cases. This allowed the researcher two triangulate the data with final written reflections and the final interviews. Only four questions were created prior to the focus group interview, which allowed ample time for the PSTs to initiate and introduce some topics of conversation, but essentially to elaborate and expand on topics introduced by the researcher. The core focus group questions were centered on the two research questions, although themes and categories identified in the final written reflection and interview were incorporated as well. These general questions became more specific as PSTs discussed their shared experiences. As the moderator, the researcher kept the conversation on topic while still allowing the freedom of expression that characterizes successful focus group interviews (Krueger & Casey, 2000).

The second reason for choosing focus group interviews was to use them as member checks. Preservice teachers were presented with initial themes and categories and were then able to give feedback, check for accuracy, give clarification or expand information, and confirm or disconfirm the researcher's conclusions up to that point in the study. Using the focus group interview gave the PSTs the opportunity to examine and revise emerging themes, greatly adding to the strength of the analysis and the overall rigor of the study (Morrow & Smith, 2000).

The focus group interview was conducted in a comfortable setting (the same office used during one-on-one interviews, which has multiple tables and chairs for the group to sit and discuss) to encourage a sense of support and an environment of sharing. The focus group interview took approximately ninety minutes to complete. The PSTs were allowed to speak as much as they desired on each topic in order to get rich, descriptive data (Creswell, 2007). The interview was digitally recorded and the researcher took notes with the smart pen, as well.

A six-member panel of experts read and approved the questions used for the one-on-one and focus group interviews prior to the study. The panel of experts are all professors. Three of them are experts in qualitative research, two of them are both mathematics education experts with qualitative research experience, and the last one is a mathematician and mathematics educator. Suggestions were made to improve the interview questions. One suggestion was to use the word "describe" instead of "reconstruct" when asking the interviewee to go over what happened while working with the child. Another suggestion pertained to the structure of the following question: When I say "learning trajectories" and "assessment", what comes to mind? The concern here was that the interviewee might not know if the intent of the question is to have them discuss each term independently or together. The intent of the question was to have the interviewee consider them together. The question, and ones similar to it, were rewritten for clarity.

Data Analysis

The data analysis strategy for this study specifically followed Stake's (2006) methodology for collective case study analysis. Essentially it was a series of identical individual case analysis for each case undergoing data immersion and description, direct interpretation (initial immersion), categorical aggregation (looking for themes), establishing patterns, developing naturalistic generalizations, within-case analysis, and cross case analysis (Stake, 2006). The analysis intended to identify and define patterns and themes relevant to the PSTs' use of LTs during the tutoring project and, subsequently, their reflection on their use of LTs and their plans for future use.

Single case. Stake (1995, 2006) stated that the first priority of a collective case study analysis is to do justice to the specific cases (i.e., to do a good job of "particularization" before looking for patterns across cases). A case analysis was conducted one at a time, completing one before moving on to the other. Each case analysis was conducted using identical methodologies.

The start of the data analysis involved the researcher immersing himself in all of the data in its entirety, multiple times. This allowed the researcher to begin developing a holistic sense of the data (Stake, 2006). The first immersion was transcribing the audio recordings from the interviews and observations for a single case. Next, the researcher read through reflective journals, transcriptions from interviews and observations (along with his corresponding fieldnotes), and lesson plans in chronological order, making small notes in the margins about thoughts, ideas, or initial concepts that were surfacing. The next step was the descriptive phase.

The descriptive phase. The descriptive phase involved editing the data to make it more manageable, which further develops a holistic, thick, and detail rich description of the case in its entirety. Stake refers to this as a case record (Stake, 2006). Data that was meaningless to the research questions were removed, resulting in greater clarity and detail. Specifically, any topics unrelated to assessment, planning, instruction, and/or LTs were cut and pasted into a "junk" document. The researcher kept this document in a separate folder just in case he needed access to if for some unforeseen reason. But this never happened. However, it set the stage for later analysis, as well as giving the researcher a better sense of the whole (Stake, 2006). The remaining data were copied and pasted into a blank *transcript form* document (see Appendix G) into the left column, leaving room for coding in the right column. All data for an individual case was kept together. However, each data source and its data was put into separate transcript forms and labeled. For example, all three interviews with Beth were placed into three different transcript forms. They were then put into a folder labeled interviews. And that folder was inside another folder for Beth. This process was followed-up by direct interpretation, which began the coding phase of the study. The next section gives a brief overview of coding and its process.

Coding. Codes are often defined as ‘‘tags or labels for assigning units of meaning to the descriptive or inferential information compiled during a study’’ (Miles & Huberman,1994, p. 56), and are the initial step in analyzing qualitative data. To ensure meaningful tags or labels, codes are assigned to chunks of data, such as phrases, sentences, or paragraphs connected to a context or setting (Miles & Huberman 1994). Codes can be developed a priori (theory-driven), through open coding (data-driven), or they can grow from a project’s research goals and questions (structural); the most common being theory or data driven (Miles & Huberman, 1994). Theory-driven codes typically require constant revisiting of theory, where data-driven and structural codes require repeated examination of the raw data. Regardless, developing codes is an iterative process. This study used data-driven codes so that themes could emerge from the raw data, but also used theory-driven codes supported by constructivist learning theory.

Open and axial coding. Corbin and Strauss (2008) identified two major levels of coding: open coding and axial coding. When starting to code qualitative data, step one is to engage in open coding or ‘‘breaking data apart and delineating concepts to stand for blocks of raw data’’ (Corbin & Strauss 2008, p. 195), allowing for exploration of the opinions and meaning within the raw data.

Once codes are created through open coding, they are analyzed through the process of axial coding. That is, coding which enables the researchers to identify connections between codes. In this study, open coding took place during the direct interpretation phase.

Direct interpretation phase/open coding. In the direct interpretation phase, the researcher pulled apart individual instances within the data and reassembled them to draw meaning, while being careful to look for multiple instances (Stake, 2006). The researcher highlighted quotes, labeled them with potential themes, and wrote brief comments (see Appendix H). The main strategy here was to highlight important information in the transcript in yellow along with a number in parenthesis. The idea is to have these direct quotes ready for the researcher to use in chapter 4. Also, in the right column, the research used red for main codes (possible themes) and blue for main descriptions of that code or possible sub-codes. It was too soon to know at that phase what codes may or may not end up developing into.

Categorical Aggregation/axial coding. The next analysis phase was to conduct categorical aggregation, in which the researcher combined individual instances into categories of similar meaning relevant to the research question (Stake, 2006), enriching the data in regards to the purpose of the study. At the same time, axial coding was taking place as the researcher looked for connections between codes. During this phase, categories and codes were allowed to reveal themselves naturally. Once the researcher found categorical themes, he placed them into a document with color-coded hierarchical themes and codes (see Appendix I). The four main color codes were assessment, planning, instruction, and future use. Mini hierarchical themes, subthemes, and codes were created.

Even though the researcher tried to set up questions in reflective journal prompts and interviews to keep the data more manageable, he could not control where the PSTs went with their discussions or reflective journal entries. This caused quite a challenge, but it forced the researcher keep viewing the codes through the four main categories mentioned. Once a code was determined to fit in one of the four categories, further decisions had to be made. The final phase in the individual case analysis process was the within-case analysis.

Within-case analysis phase. During the within-case phase of analysis, emerging themes were more closely defined by identifying connections between and among them. In addition, themes were illuminated by connecting verbatim passages and direct quotes taken from the data (Stake, 2006). This required the researcher to pull patterns and themes from the different data sources to narrow them down to overarching patterns and themes. For example, a new document for assessment was made. If a code for assessment was found during the previous phase, then a determination had to be made if the code represented PSTs using LTs for assessment, reflecting on their use of LTs for assessment, or if they described future use of the LTs. These themes and codes intertwined in a myriad of ways, forcing the researcher to experience his own trips through assimilation and accommodation. Ultimately, though, by combining the patterns, there was convincing evidence for the interpretations of the data for each case.

The aforementioned process was conducted for each of the individual cases in this collective case study. Each case was considered separately from the others. As stated by Stake (2006), doing justice to the individual case is of utmost importance. Once that occurred, then cross case analysis began.

Collective cases. In the cross-case analysis phase, the researcher developed a thematic analysis across all the cases, comparing and analyzing the individual case patterns and themes, and found commonalities. From this, interpretations, called assertions, of the meanings of all the cases were made (Stake, 2006). This required the researcher to create further hierarchical themes and codes using the same color-coding system as the single case, categorical aggregation phase (see Appendix J). In order to keep the data and the data sources organized, if a PST contributed a specific theme or code, then the first letter of their name was placed in the box. Next, themes and codes were further combined until a theme emerged that came from multiple sources within a case and from multiple cases.

The final phase of data analysis process is the definitive focus of this qualitative study: the interpretive, or lessons learned, phase. In this phase, the researcher developed naturalistic generalizations to define the meaning of the cases (Stake, 2006). These lessons learned are what the reader learns from the collective case study (Stake, 2006). Stake (2006) extends the naturalistic generalizations to include the kind of learning that readers take from specific case studies. The "lessons learned" are presented in chapter 4 with detailed descriptions of the themes, direct quotes from participants, and the researchers interpretations.

Summary. The intent of data analysis was to develop a deep and rich understanding of cognitive processes of PSTs as they used LTs throughout the tutoring project and how they reflected upon their use of LTs and how they might use them in the future. To do this required that the data be gathered together and organized into a sequence that tells the story as the PSTs intended to tell it (Stake, 2006). The following data collection tools were analyzed in order to tell the PSTs' stories: reflective journals, lesson plans, observations, and interviews. Figure 3 is an outline of these questions and their corresponding data sources.

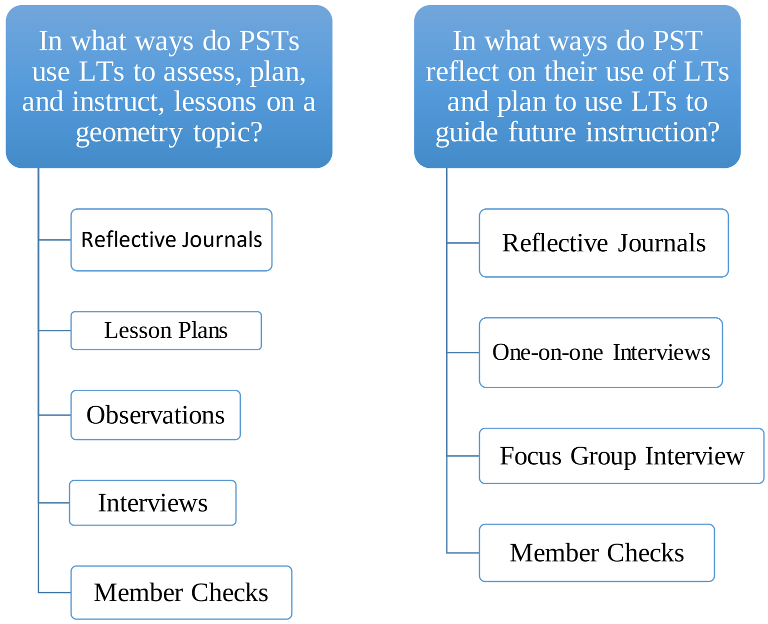


Figure 3. Data sources for each research question.

Trustworthiness

In order for the reader to trust the findings, Merriam (1998) recommends multiple strategies to enhance the trustworthiness, allowing for stronger similarity between the participant's reality and the researcher's interpretation of that reality. This study used triangulation of data, member checks, and a panel of experts to review the interview protocols, along with the validity of the tests used for the study.

First of all, thick description are provided in the results chapter for the reader to understand the setting, PSTs, and data analysis in more clarifying ways. The descriptive nature of case study allowed for an end result of a "rich, thick description of the phenomenon under study" (Merriam, 1998, p. 29). This description includes many variables and how they interact over a period of time. This way readers will be better informed if they want to replicate this study or interpret their own findings in a similar study. In order to use reliable data, triangulation of data and members checks were used to determine what results should be used.

In order to triangulate the data during the analysis phase of the study, collective case studies need to have at least three data sources to assure that key meanings are not being overlooked (Stake, 2006). Triangulation is the "process of repetitious data gathering and critical review of what is being said" (Stake, 2006, p. 34), and attempts to assure that accurate information and interpretations are obtained. It also serves to clarify meaning by using different methods to see the case (Flick, 1998; Silverman, 1993). In order to triangulate the data, this study used reflective journals, lesson plans, interviews, and observations. Throughout this process, member checks were used, which Stake (2006) states is another way to ensure trustworthiness. Throughout the duration of the study and each phase of analysis, both within-case and cross cases, PSTs participated in member checking in order to assess the researcher's interpretations of their views, as well.

Lastly, reliable tests and materials must be used in order to increase trustworthiness of the findings. For example, The van Hiele test, which was used to employ maximum variation sampling, was found to be a reliable test (Usiskin, 1982, 1990). Battista's (2012) *Cognition-Based Assessment & Teaching of Geometric Shapes* (CBA) was used as a diagnostic tool for the clinical interview. These materials have "gone through extensive filed testing with both students and teachers [and is] consistent with major scientific theories describing how students learn mathematics with understanding." (p. xvi).

Limitations

A limitation of this study was the selection of PSTs. By selecting only PSTs in the capstone course (at that point they were approximately one year away from graduating), the researcher limited the possibility of understanding how preservice teachers in other, earlier, courses might be impacted by this study.

Another limitation of this study was that the researcher selected the LT assessment tool rather than providing students with options. The tool for this study had very specific goals and objectives in mind to help students increase their understanding of LTs. It is possible that other LT assessment tools may have been better suited and may have had different results.

A final limitation of this study may have come from the researcher's relationship with the PSTs. Because of prior courses taught, the researcher had the opportunity to build rapport with the PSTs. It is possible this rapport influenced them to participate in the study. Similarly, it is also possible that once they agreed to participate, they may have been more serious about their work because of our built rapport and not wanting to disappoint the researcher.

Conclusion

This collective case study provides rich and thick description of preservice teachers' use of learning trajectories during a tutoring project, as well as reflections on their use and how they might use LTs in their future instruction as a result of the project. The researcher hopes to add to the research literature by providing ideas about ways teacher education programs might better prepare PSTs to deal with the challenges they will face in their future classroom.

CHAPTER 4: FINDINGS FROM DATA ANALYSIS

Introduction

As mentioned in Chapter 3, the intent of this data analysis was to develop a deep and rich understanding of the cognitive processes of PSTs as they used LTs throughout a tutoring project, how they reflect upon their use of LTs, and how they might use them in the future. To do this required data to be gathered together and organized into a sequence that tells the story as the PSTs intended to tell it (Stake, 2006). The research methodology for this study was a collective case study which focuses on a single or specific issue or concern within bounded cases (Stake, 2006). In this study that issue was the use of LTs during a one-on-one tutoring project. Three cases were selected for this study, and then later compared for patterns and themes during data analysis (Stake, 2006). The data analyzed in this study were reflective journals, lesson plans, observations, and interviews. Finally, data and theory-driven coding was used to interpret raw data and create themes. With this purpose in mind and supported by the study's constructivist learning theory as the theoretical framework, responses to the following research questions were examined:

1) In what ways do PSTs use LTs to assess, plan, and instruct lessons on a geometry topic?
2) In what ways do PSTs reflect on their use of LTs and plan to use LTs to guide their future instruction?

In this chapter, the findings of the study for each question will be presented as themes and supported by data. For research question one, the following themes emerged: 1) For the use of LTs for assessment - PSTs showed flexibility in identifying their students' level of thinking about shapes; 2) For the use of LTs for planning – PSTs planned lessons within their student's *zone of proximal development* and created lessons that encouraged active learning; and 3) For the use of LTs for instruction – PSTs used LTs as a formative assessment tool through active learning. For research question two, the following themes emerged: 1) Reflection on use - PSTs felt the LTs helped them create and use a *cognitive road map* of their student's thinking; and 2) Reflection on future use – PSTs considered LTs as a tool for *hetero-homogeneous* small-group instruction and for *building up* during whole-group instruction.

Research Question 1 Sub-Question: Assessment

- In what ways do PSTs use LTs to assess for lessons on a geometry topic?

Learning Trajectories and Assessment

Teachers must attend to students' prior knowledge in order to determine how students are likely to attend to tasks. It seems natural then that a teacher must first engage in assessment before moving to the next phase of the teaching cycle: lesson planning. Teachers might blindly choose instructional goals or follow their curriculum using activities their students may not be ready for. According to the National Council of Teachers of Mathematics' *Principles to Action: Ensuring Mathematical Success for All* (2014), "Effective teaching of mathematics uses evidence of student thinking to assess progress toward mathematical understanding and to adjust instruction continually in ways that support and extend learning" (p. 10).

Theme: flexibility in identifying students' level of thinking. Learning trajectories hypothesize levels of sophistication the mind goes through over time as it wrestles with mathematical ideas. Each level on the trajectory tells how a child at that level understands the mathematical concept and gives examples of what that might look like.

The goal of the clinical interview was to identify where the student's level of thinking falls on the learning trajectory construct. While the PSTs were given an assessment tool with questions that addressed the LT levels, they were not instructed on how they should use the results of the assessment to identify their student's level of thinking. There was some preliminary apprehension toward "labeling students" or "putting students in boxes," but that eventually lead to the PSTs seeing flexibility in their use.

Beth showed clear signs of struggle when determining a level for her student. She asserted that:

> Determining his level has proven to be quite a challenge for me…there were some minor, but key, points he was missing in his statements about shapes and their rules…There are a few gaps in what I saw of his thinking.

Later, she discussed her thoughts after spending a day reflecting on the assessment.

> After much debate and coming back to this numerous times over the last 24-ish hours, I've decided to place [him] at a level 2.2 (see Appendix K) I am not 100% confident in this, but then, I guess any time you try and put someone in a particular "box" they never completely fit.

One key point she makes here is about trying to "put someone in a particular box" and how they will never completely fit. It was her decision to determine how to use the data from the interview to determine where the student's level of thinking is based on the LTs. More importantly, she decided to place him in only one level even though she was not 100% sure he belonged there. Beth placed him at a 2.2, at which students use a combination of both formal and informal language to identify shapes. However, this level of thinking still relies on the visualization shapes as a whole. In the Fieldnotes for Observation 1, it was noted that she asked her student why he turned his head while looking at the triangles and other shapes (see Figure 4). In Journal 2, she talked about her student turning "… his head more than once to view shapes that were "at a diagonal," which is similar to what the CBA addresses.

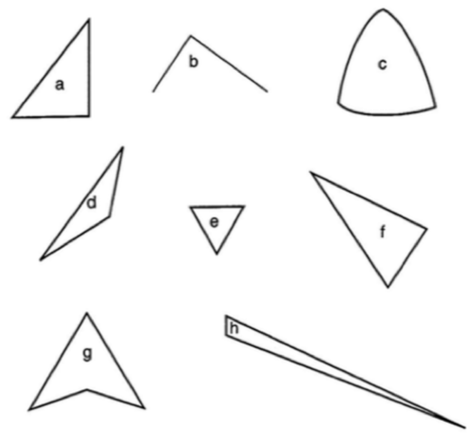


Figure 4. Question #4 Illustration from Cognition Based Assessment Tool

Afterwards, she questioned the level she originally chose and began to doubt herself again:

> I was still leaning towards a level 2.3 (see Appendix K) as I felt like he had mastered, or almost mastered, the majority of the things in the level. However, once I started reading level 2.3 … I don't know. I just don't think he's quite there yet. There are a few things in level 2.2 that I need to know he truly understands, first. I am hoping that my first lesson will help me assess that and also leave space for some wiggle room if I think we are okay to move on. (Journal 2)

She refused to say, definitively, that her student was at a 2.2 or a 2.3. She saw many traits of a 2.3 but was not confident that he had mastered 2.2. Her explanation:

> One of the places where [he] performed closer to a level 2.3, in my opinion, was in the first task with drawing machine 1 (see Figure 5). He correctly stated that the rule was that they had to have a 90-degree angle. He also correctly identified obtuse and acute angles in the following sub-problems (see Figure 6), which was more specific than the level 2.2 got. (Journal 2)

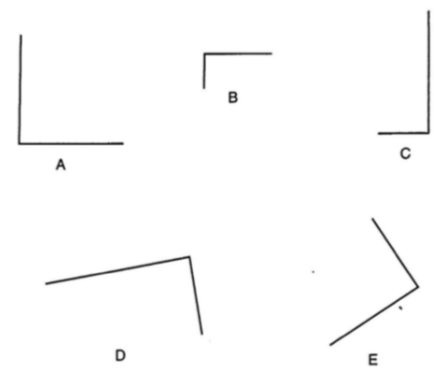


Figure 5. Image from the Cognition Based Assessment Tool: Question #1a.

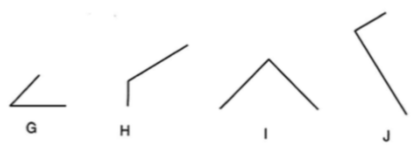


Figure 6. Image from the Cognition Based Assessment Tool: Question #1c.

This is a pivotal point in her reflection. Her resistance to assigning a level eventually lead to the determination that she does not necessarily have to choose one level, giving her the freedom to work between levels.

Melissa also dealt with similar issues when it came to choosing a level for her student, but unlike Beth, she was hesitant about predicting where her student might be prior to the clinical interview, something that the other two PSTs did not express. When asked in Journal 1 about where she thought her student might be prior to the clinical interview, she stated that "Each student can be very different from one another and I do not want to get caught up in where they are expected to be." She went on to say, "I didn't want to get caught up in what fifth graders are supposed to be able to do … I struggle so much with putting labels on children." Irrespective of her hesitation with labeling, she was willing to do so after the clinical interview. She even began showing signs of flexibility in their use.

Journal 2 was a reflection done immediately following the clinical interview; her first encounter with the student. She was asked to identify her student's level and to provide evidence to support the decision. The results were as follows:

> I would classify this child as a high 1.2 or a low 2.1. I [chose] 1.2 at first because the student was separating shapes using the terms "more slanted" and "more like a ramp than the other one" … According to the book, a level 2.1 is a student that is able to point out that there is a corner and two lines. My student did not do this exactly, but he was quick to point out that there were three "ends" to the shape (see Figure 5). When it came to [the] next question (see Figure 7) he was all about looking at how many sides the shapes had. He compared some of the shapes to diamonds and ruled out ones that had "too much slant." I do not recall him pointing out that they have four points, but he knew that there were four sides on each. This lead me to believe that he is a level 2.1 but still has some tendencies of 1.2. (Journal 2)

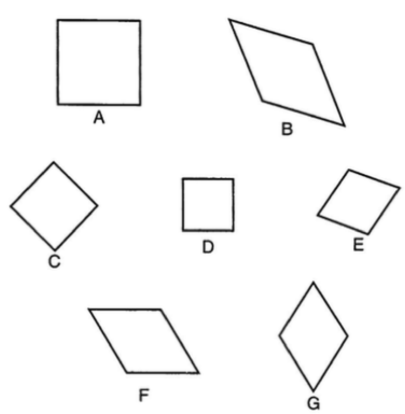


Figure 7. Image from the Cognition Based Assessment Tool: Question #2.

Not only is she considering multiple levels at one time, she is also displaying flexibility in how she interprets her student's work and explanations. Previous comments like, "My student did not do this exactly, but…" and "I don't recall him pointing out that they have four points, but he knew there were four sides… [which] lead me to believe…" are examples of this. She went on to say that her student was at the "same level as that child in the book, just he didn't use those words."

In Interview 1, we discussed Melissa's decision to choose to place her student in multiple levels and if she was okay with it. She explained that prior to the clinical interview, she was worried about being fair with her assessment and again touched on her concerns with labeling students. But after the clinical interview, she no longer saw the levels as strict guidelines. She added that you do not have to "put the child in the box of just [one] level. They can be in between two levels. They can be on their way to one level, or maybe they need to be stronger in a different level… They're very adaptable. They can be used for any child at any time."

Unlike Beth and Melissa, Nan was not apprehensive about choosing a level for her student. But like the other two, she did not feel it necessary to choose only one level to identify her student's thinking. She identified her student's reasoning as spanning three levels: 1.1, 1.2, and 2.1. In journal 2 she was asked what level her student was at and to provide evidence. Her response:

> I believe this child is at level 1.1. At times, she showed evidence of being a 1.2 or higher, but her reasoning proved only that she knew a few geometric terms, she really did not understand how to apply them nor was she consistent in their definitions. (Journal 2)

According to Battista (2012, p. 9), students at Levels 1.1 and 1.2 focus on shapes as "visual wholes." They may identify the name of a shape by comparing it to something else. For example, they might identify a rectangle on a piece of paper accurately. But when asked why it is a rectangle, they might say, "Because it looks like a door." Nan's student "knew that a rectangle must have four right angles, but she also argued that it "must look like a rectangle," too (Journal 2, p. 3). The difference between a 1.1 and a 1.2 is accuracy. Students at a 1.1 often name shapes incorrectly, while 1.2s get them correct. These students do not focus on specific properties of shapes, which leads to a 2.1.

A student at a 2.1 begins to use properties of shapes to justify the name they have chosen. Their language is informal and the meaning behind it could be misinterpreted. A student might say a shape is a rectangle because it has 4 corners. The problem with this is more complexed than using inaccurate mathematical terminology like angle and vertex. Of course we want students using precise language as they grow. Nevertheless, in this example, we are not sure what the student's definition of a corner is. It is possible that they actually mean right angles when talking about the corners. It does not necessary imply that they will look at any shape that has four corners (e.g., a parallelogram with a set of obtuse and acute angles) and identify it as a rectangle. Nan's student shows similar traits to this, which she was able to recognize.

Like Melissa, Nan showed flexibility in interpreting her student's reasoning about shapes. Acknowledging that a student at a 2.1 uses informal language, she looked for such language. However, it was not always what the student said, but also what the student did not say, that helped her identify her student's level of thinking. During Interview 1, she explained that:

> I could tell that she knew what a rectangle was even though she misidentified some of the angles, but she did notice that the [shapes] that don't all have 90-degree angles shouldn't be rectangles at all… she subtly or subconsciously recognized these characteristics though she wasn't able to articulate it. (Interview 1)

Beth had a different experience. She believed her student to be at either a 2.2 or 2.3. Since formal descriptions of shapes become paramount as the student works between these levels, having flexible interpretations gets stifled.

All three PSTs ultimately showed flexibility in identifying their students' level of thinking about shapes, coming to the conclusion that it is not necessary to label a student at only one level. This flexibility is supported by the theoretical framework in two ways. First, the PSTs themselves went through either assimilation or accommodation as the grappled with their understanding of the purpose use of learning trajectories for assessment. For example, neither Melissa or Beth felt comfortable with "putting their student in a box," which identifies their initial schema, or beliefs about how the results from the clinical interview should be use. However, this schema eventually found equilibration for a new schema of flexibility.

In addition, from the PSTs' perspective, they created new schemas of understanding that LTs provide not a fixed, single point of understanding from their students, but rather, the levels assigned where a zone of proximal development form which they could work within. Not only were they flexible, but they were also able to defend their choices with ample evidence from their students' thinking. This same flexibility was seen in both planning and instruction, which will be discussed throughout the remainder of this chapter.

Research Question 1 Sub-Question: Planning

- In what ways do PSTs use LTs to plan for lessons on a geometry topic?

Learning Trajectories and Planning

An important concept for this study, Vygotsky believed that learning takes place when children are working within their zone of proximal development (ZPD), which is the gap between the actual developmental level and the level of potential development under adult or peer guidance" (Bruner, 1986). In order to build upon students' mathematical knowledge, it is important to consider the gap between the students' starting point and the big idea of the mathematics being taught. That is to say, when planning lessons teachers ought to consider how to meet students at their starting points and build them up from there based on what next steps are developmentally appropriate. Mathematics educators have noted, in recent years, the importance of instructional planning when the goal is to build upon students' current mathematical knowledge (Gravemeijer, 2004). In this study, aspects of the PSTs' lesson plans exposed a cognizant effort to build upon their students' thinking and to stay within their students' ZPD. More importantly, there were similarities and differences across, and between, the cases in how the PSTs planned to work within their students' ZPD.

Theme 1: planning within students' zone of proximal development. Preservice teachers were asked to create lesson plans after their initial assessment and each subsequent lesson. They were not given a formal lesson plan template to fill out. They did, however, submit a lesson plan. In addition, they were asked to journal about their plans, and these plans were discussed during interviews. This study found that the PSTs used the assessment data and the trajectories to decide a range of levels, or ZPD, they wanted to focus on in their planning. This was not surprising in a one-on-one environment. It was the way in which each PST planned to work within the students' ZPDs that was most intriguing and will be the main discussion of this theme.

All three PSTs used the results from the assessment data gathered in the clinical interview as the motivation behind the learning goals determined during the planning phase of this study. As mentioned in the sub-question on assessment, the data revealed that the PSTs showed flexibility in identifying students' level of geometric thinking. Meaning that they did not feel it was necessary to identify their students at only one level. For example, if a student showed evidence of understanding elements of both levels 1.2 and 1.3, then the PSTs felt comfortable placing them within both the levels. They showed similar flexibility when determining what learning goals they wanted their lessons to focus on.

It is important to distinguish two things before moving forward. First, when referring to the planned goals of the lessons, the intent of this section was not to compare the specific content of the goals in and across the three cases (i.e., all three PSTs focused their goals on teaching about parallel lines). Instead, the following discussion looks at what LT level(s) the lesson plan goals fall under and how that compares to where the PSTs placed their students' level of understand after the clinical interview (e.g., if a PST placed their student at a level 1.1 after the assessment, did they decide to focus their learning goals strictly within the next level up, 1.2?).

Second, it is important to clearly define what level the students showed mastery in to compare the similarities and difference between what levels the PSTs chose to focus on when determining their lesson goals. After the clinical interview, Nan, for example, placed her student at a level 1.1 because she felt her student was proficient with all the concepts within that level and was ready to move one. She felt her student had shown elements from levels 1.2 and 2.1 but was not proficient in either one (Journal 2). Conversely, Melissa chose levels 1.2 and 2.1 as the levels her student was in, even though her student had not shown complete proficiency with either one (Journal 2). Her student, like Nan's, was proficient at level 1.1. Beth, took the same approach as Melissa. She chose to focus her attention on level 2.2 as a lens for creating her lesson, even though her student really had only grasped the concepts up to level 2.1 (Journal 2). The importance of this distinction will become apparent throughout the remainder of this discussion as it dives into the range in which the PSTs planned to work beyond the highest LT level each student was already competent with. Ultimately, regardless of what level the students were identified as most proficient in, each PSTs used that level as a starting point for their lessons and created activities that were within their students' zone of proximal development.

The PSTs were not told to work within the students' ZPD, nor what LT levels fit it. However, it seems only natural that the PSTs would plan to work with the next level up from the highest LT level each student was proficient in, which all three PSTs did, to some degree. Battista (2012) recommends having students work on several problems in the next level before moving them to the next level.

Melissa (Journal 3) felt her student showed some competence, but not all, in levels 1.2 and 2.1, which are the next two levels above his level of proficiency. When deciding what her goals were moving forward, she stated that:

> The student is on the edge of being a 1.2 and a 2.1, meaning that they need further instruction on how to describe shape properties. They need this to be able to eventually progress into connecting property-based characteristics and formal geometric concepts. (Lesson Plan 1)

At first glance, one might assume that her student needs further instruction with 1.2 in order to get over the threshold to 2.1. However, she is actually stating that the student needs further instruction in both of the levels, simultaneously. When she says, "They need this to be able to eventually progress into connecting property-based characteristics and formal geometric concepts" (Lesson Plan 1), that is a direct reference to the assessment book which describes what a student can do at a level 2.3 (Battista, 2012). Not only is Melissa planning to work with concepts that are one and two levels up from her student's level of proficiency, but she is also keeping in mind where all this will lead to further down the road.

In addition, she said:

> I made sure that I knew some of the key terms from the lower one [1.2] and the one immediately above it [2.1] because he was already kind of on the edge of two levels

> anyways so I wanted to make sure that wherever I was going, I was progressing towards the one that was above where we were, and if he said some words that were from the one below, I wanted him to feel comfortable enough to progress those into something more. I wanted him to have a full understanding before moving forward with that. (Interview 3)

Interestingly, her focus is getting the student to a 2.1, not to become proficient at 1.2 and then move to a 2.1. She makes her focus on the highest level the student has within his grasp, while simultaneously incorporating missing pieces from the previous level.

Nan's planning goals showed some similarity to Melissa's. Her student, like Melissa's, was proficient at a 1.1 and also showed some understanding in both levels 1.2 and 2.1 (Journal 2), and she planned to work between the two levels above (Journal 3). But Nan's purpose for the two levels was quite different than Melissa's. Nan stated that:

> Even though the lesson was two levels above [2.1] her at some times, the goal was really one below [1.2], but she had to learn to take that step because she's being pushed a little bit harder. She had to take that step into the goal level to be able to work with what the lesson was at. Though she didn't understand the lesson per se fully, that was okay because that's not my point. My point was to get her into that level just below where the lesson was at. (Interview 3)

Nan planned to work between two levels (1.2 and 2.1), hoping that by pushing her student with concepts from 2.1, it would help her student become proficient in 1.2 (Journal 3). Her goal was not to have her student become proficient in 2.1. Melissa did the opposite. She planned to work between the same two levels as Nan. But unlike Nan, she focused on getting her student to a 2.1 and hoped that in doing so, her student would pick up some of the missing pieces along the way (Journal 3).

Beth was the only one of the three PSTs to focus her lesson plans one level up, 2.2. Similar to Nan, she did recognize that her student showed understanding two levels above and she also focused her goals on the student's next level up [2.2], but she was not convinced that she should do any work two levels up [2.3] or more (Journal 3). Prior to this, in Journal 2, Beth mentioned that:

> I am also seeing a few more areas where he is lacking [in 2.2]…He struggled with differentiating between parallelograms and rectangles, could not identify that the mix of squares and rhombi all had equal sides, and he turned his head more than once to view shapes that were "at a diagonal." (Journal 2)

Another reason was the student's lower level of proficiency with the use of language. Beth explained that:

> I'm trying to get him to that point to correctly classify [names of properties]. I feel like the focus is mostly on the language. I haven't really looked at that as much because he has the vocabulary. I'm kind of, in a way, trying to help him apply that vocabulary. (Interview 2)

Beth felt it was necessary to develop her student's use of language prior to moving on to the complexity of levels beyond 2.2. The focus on language was unique to Beth. This is not surprising, though, because her student's level of proficiency was two levels higher than Melissa and Nan's students. As a student progresses along the trajectory, there is a transition between the use of informal to formal language. Beth's student had reached a point where formal language is necessary to work within the next level, which explains her reason for planning to work only one level higher.

Melissa and Nan both created lessons that stayed within their students' ZPD by planning to work with concepts no higher than two levels above their level of proficiency. And while their reasoning and approaches were unique to each other, there seemed to be no desire by either of them to work beyond the two levels, which seems to be evidence that they wanted to stay within their students' ZPD. Beth, on the other hand, felt she could only work one level above her student's level of proficiency, but only because the necessity for accurate use of formal language becomes paramount at the higher levels.

Theme 2: lesson plan activities that encourage active learning. Pedagogical approaches that promote active learning or "learning by doing" are at the heart of constructivism. Active learning is largely defined as any instructional method that engages students in the learning process, requiring the student to do meaningful learning activities and think about what they are doing. The contrast to this is traditional lecture-style or teaching-by-telling where the student passively receives information from the teacher. Active learning leads to more meaningful learning (Freeman et al., 2014; Henderson et al., 2010); which has been shown to increase student success and engagement in the classroom (Freeman et al., 2014).

Active learning is an essential element to meeting the needs of children. The National Council of Teachers of Mathematics (NCTM) has long promoted pedagogical methods that require students to be actively engaged in forming new knowledge with conceptual understanding (NCTM, 2000; NCTM, 2014). Instructional strategies centered on active learning include problem-solving tasks, questioning, and inquiry. The PSTs centered their lesson plans around the students being actively engaged.

All three PSTs chose activities that encouraged the students to be actively engaged in the learning process. When discussing the activities in her first lesson, Beth wanted to:

> … develop a concrete "map" (see Figure 8) of the student's current thoughts on the properties of various shapes, as well as being able to look back and see how his thoughts

> have progressed. In *exploring* individual shapes and stating the observed properties, the student will be progressing toward a level 2.3 (see Appendix K) in which he will be able to "formally describe shapes completely and correctly." … In laying out the properties associated with the shapes, and *allowing him to make discoveries/conjectures on his own*, he will hopefully move past identifying shapes beyond just their appearance and instead pay more heed to the properties. (Journal 3)

The anchor chart not only gives her and the students a concrete map of his thinking about shapes, it also gives Beth a chance to examine the ways in which he describes the parts of the properties of shapes. In her last quote, Beth discussed that her goal is to move her student to a level 2.3. These students describe all the properties of the shape both accurately and formally. For example, the formal and accurate definition, according to Battista (2012), is that a rectangle has "opposite sides equal and parallel, and 4 right angles" (p. 33). The anchor chart will work as an on-going assessment tool, allowing Beth to identify any shapes that her students identify at 2.2 (combination of formal and informal descriptions) or even a 2.1 (informally describes parts or omits properties of a shape). Beth did not feel like she has gathered enough data about the student and this point, so the anchor chart allowed her to determine his level more accurately after the lesson.

Beth emphasized giving the student time to think about and discover the properties of the shapes while making the anchor chart, opposed to her telling him. And while it was not relevant to active learning, it is interesting that she mentioned that both her and her student would be looking back to see how his thoughts progressed, which is a sign that she is considering metacognition in her lesson planning. It was relevant that she was considering how he was creating new schemas, which fits the constructivist learning theory.

In order to make the concrete map of her student's thoughts on various shapes, she created an anchor chart (see Figure 8) so he could tell her "all of the things he knows or notices about the various [quadrilaterals]... and… put what he says about each shape on a post-it note and place it with the shape" (Lesson Plan 1).

For instance, if he said a rectangle has "four 90 degree angles," then that would have been written on the post-it note and placed in the rectangle section. This would allow him to continue to be actively engaged moving into the next activity.

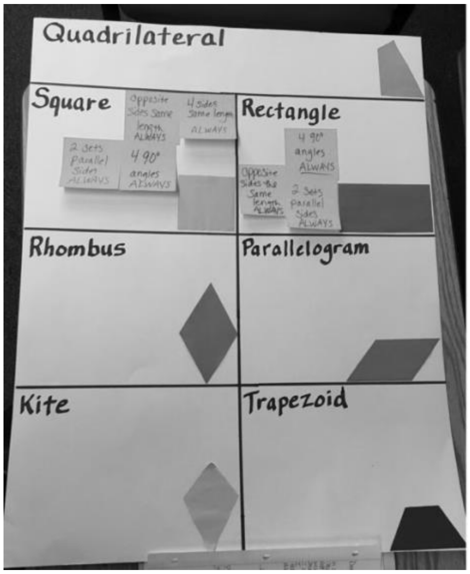


Figure 8. Beth's anchor chart with some square and rectangle properties identified.

After finishing the first phase of the anchor chart, she wanted her student to "begin exploring all the different quadrilaterals using *The Geometer's Sketchpad*, starting with the square. "*As he drags the shape around and explores it, I will ask him what he notices*" (Lesson Plan 1). The Geometer's Sketchpad is a dynamic software program which allows "… students to explore draggable geometric constructions. A construction is draggable if it maintains its geometric properties when its vertices or sides are dragged with the mouse" (Battista, 2012. p. 66). For example, when exploring the rectangle maker, the shape starts out as a "traditional-looking" rectangle (see Figure 9). It is "traditional-looking" because many rectangles students see have the longest side as its base, running horizontally, while the two shortest sides run vertically. But upon further review of it, the user discovers that not only can its orientation change (see Figure 10), but all four side lengths can be congruent (see Figure 11). The point of this is to discover the properties a rectangle always has.

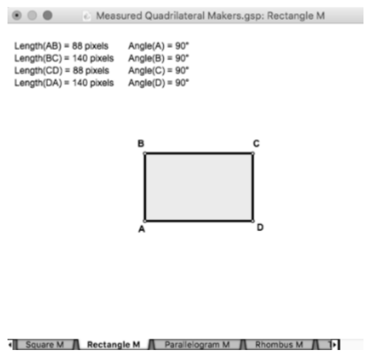


Figure 9. Traditional-looking rectangle.

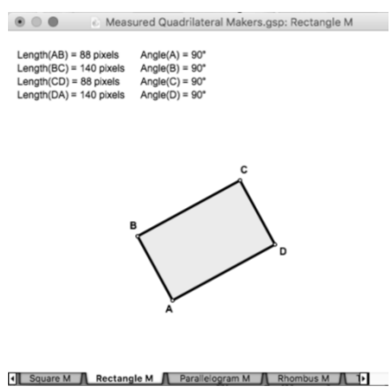


Figure 10. Slanted rectangle.

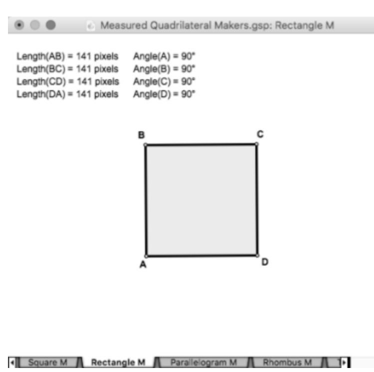


Figure 11. Rectangle with four congruent sides.

This activity not only lets the student explore and discover the properties of the shapes, it also addresses another issue Beth mentioned about her student's level of thinking: measurement. She stated that:

> If necessary, I will ask him leading questions (can you make a rectangle … what do you notice about its sides). I will add the conclusions he draws to our board. If there is a contradiction between what the student said prior to using The Geometer's Sketchpad and his most recent conjecture, that post-it will be covered up, but still left on the board so that it can be referred back to. We will go through each shape, adding to the anchor chart as we go. (Lesson Plan 1)

She continued to plan this interplay between the anchor chart and the software program throughout the remainder of the study. In Journal 2, she mentions that, "in order for [him] to move onto a level 2.3, I need to see him show me that he can understand measurement and apply it to the properties of shapes" because measurement is a crucial element of a level 2.3. She needs to:

> … see if he is able to focus on line lengths for squares and rhombi. I also want to see if he focuses on the measurements (sides and angles) provided by the program, at all. This will give me a better idea of whether he finds these properties important when viewing shapes and how heavily he relies on how they appear, visually. Also, based on his statement that "a square is a rectangle, but a rectangle is not a square" during the interview, I want to see if he can justify this statement using what he sees in the program. (Lesson Plan 1)

By having the student drag, explore, and record the conclusions he came to, while also looking for contradictions and validating his ideas, the student was involved in active learning. But it also demonstrates her focus on doing ongoing assessment. She validates her focus of ongoing assessment further by saying that she is, "starting with him dragging the various shapes as a form of further assessment to gain further insight as to where the student is at" (Journal 1). This is reiterated in Interview 1 when she says that her remaining lessons are going to be a "continuous assessment, because how else would you know if he's moving forward in his levels if you're not constantly assessing where he's at." For Beth, this ebb and flow between technology and hands-on materials, along with her focus of ongoing assessment, continues throughout all of her lesson planning.

Similarly, Melissa planned to "… *encourage discovery-based learning*" by trying to "step back and not instruct too much." She wanted the student to "*explore on their own so* that they would have a better understanding than [she] could spoon feed them" (Journal 4). And just like Beth, she chose to use The Geometer's Sketchpad. However, Melissa was the only PST to use this tool, exclusively, throughout the entire study. Melissa's discovery-based approach fits more with Vygotsky's views, however, active learning is still at the heart of what is taking place.

In Journal 2, Melissa described her student as "on the edge of being a 1.2 and a 2.1, meaning that [he needed] further instruction on describing [parts of] shape properties." Specific to her student, he "was [identifying properties of] shapes using the terms 'more slanted' and 'more like a ramp than the other one,'" when referring to parallelograms compared to rectangles, for example (see Figure 7). Moreover, students at a 2.1 typically recognize, at the very least, that an "L" shape, no matter its orientation, "has a corner and two lines" (Battista, 2012, p.20). Her student could only "point out that there were three "ends" to the [L] shape," with no mention of angles even when asked if he recognized anything else about the shapes (Journal 2). In order to address this, her goal for Lesson 1 was to:

> … focus on identification of shapes, properties, and justification. I want to help the student understand different properties of shapes and how to use that knowledge to identify them. I plan to do an activity that has the student defend [his] reasoning behind identifying shapes… (Journal 2)

She used the same quadrilateral maker activity as Beth. She wanted him to have the freedom to *play with the software program*, and through the interaction, look for patterns and tell her the things he noticed as he dragged the shapes. She planned to start exploring with the square maker and have him work his way through all of the quadrilaterals, all the while making sure she focused some of her probing on angles of shapes to see if he would recognize them as a necessary property (Journal 2).

Identifying and working with a student's thinking at this level can be tricky and require careful interpretation of the evidence since formal geometric terms are not necessary for the student to be exploring at a 2.1. In fact, students at a 2.1 informally describe only parts of properties of shapes, not all of the properties (Battista, 2012). Battista (2012) explains, for example, that a student may "say that squares and rectangles have 'square corners'" (p. 20), which provides evidence that the student recognizes angles, and maybe even the similarity between the angles of the square and rectangle. But because the term square corners is not a formal term, their understanding of angles must still be interpreted (Battista, 2012).

Although Melissa's student did begin to show signs of recognizing similarities and differences between the properties of squares, rectangles, and parallelograms, in lesson 2 she planned to have him continue to explore the properties of individual shapes with the software program as she felt he needed to "continue to explore and identify properties of the other quadrilaterals before moving on to higher [LTs]" (Journal 3). This time, however, she used a different activity called *Predict and Check* (see Figure 12) to both validate and increase his identification and understanding of shape properties. Her goal was a continuation of the aforementioned goal and was connected to the same LTs. “Having other static shapes on the screen while moving the shape that is draggable might get him to further see the properties of various shapes and use those to justify what shapes must have. He may even begin to see differences in what types of angles and sides different shapes have, and start comparing and contrasting them, but that is not my plan” (Journal 3).

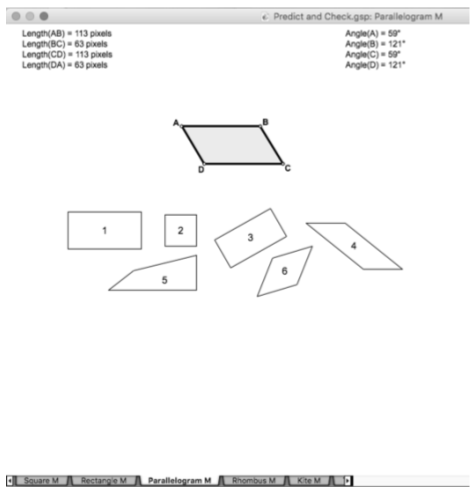


Figure 12. Predict and Check activity.

With the Predict and Check, the user is shown 6 different quadrilaterals: a rectangle, square, a “tilted” rectangle, a “tilted” parallelogram, a quadrilateral with no congruent sides, and a “tilted” rhombus. Above the shapes is a traditional image of whatever quadrilateral the user chooses. In Figure 12, for instance, the top shape is a parallelogram. This parallelogram is draggable like the previous activity and is a geometric construction which means it maintains its geometric properties. The goal of the activity is to first predict which of the six shapes the parallelogram can cover identically when manipulated. The user might think that the parallelogram can only cover #4 because it still looks like a parallelogram when thinking about it as a visual whole.

Next, the student *checks* their predictions. The student actively engages even further by dragging the shape over each figure and manipulate its vertices to see if it can cover each. Upon further review, the student would realize that the parallelogram can cover the first rectangle (see Figure 13). In fact, it can cover all the shapes besides #5.

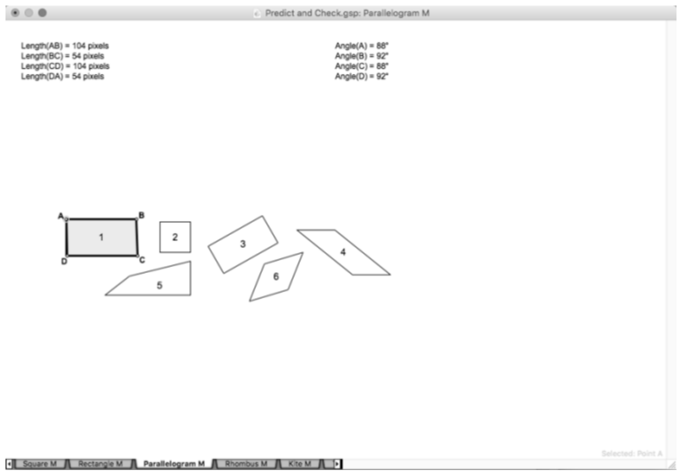


Figure 13. Predict and Check activity with parallelogram covering a rectangle.

While this activity was designed to explore the properties of each type of shape and then discover whether or not it can cover other shapes, that was not Melissa's purpose for using this activity. This activity was chosen by Melissa because she wanted her student to continue to look for parts of the properties he believed each shape must have. And, if need be, the activity afforded him that opportunity of taking what he already knows, looking for patterns in the shapes that may contradict his current understanding, and then justifying any new knowledge. For example, he believed that rectangles are "mushed" down squares, meaning he thinks that rectangles have two long sides and two short sides. Melissa thought that maybe "after exploring the rectangle, he will realize that he can cover the square in #2, contradicting his current belief that all rectangles are mushed down squares" (Journal 3). However, as previously stated, this would have been an extension of the lesson if the student recognized patterns between shapes when manipulating them (Lesson Plan 2).

In the aforementioned activities, Melissa's student is actively engaged in his learning. Not only is he "learning by doing" when he drags the shapes and looks for patterns, but he is also starting a journey through geometric proof, which will be discussed in the section on instruction. Nan also wanted her student to have an active role in the learning:

> I plan on bringing paper cut outs [she ends up bringing plastic ones] of many different shapes [examples and non-examples of polygons] - many will be of the same shape but in different sizes and "looks" if applicable (i.e., a right trapezoid and an isosceles trapezoid). I will use a white board *to allow the student to organize the shapes in whatever way she sees fit*, whether that be in lists, [plastic circles to use as] Venn diagrams (see Figure 14), tables, etc. The *student will be leading the activity and the conversation*. My only input will be asking clarification questions. For example if they say "these are all the shapes with an acute angle," I will ask "do all of these have just one acute angle or are they allowed to have more?" (Lesson Plan 1)

Phrases like "allow the student to organize," and "The student will be leading the activity" is a clear indicator that active learning is at the forefront of her mind when planning.

Figure 14. Example of Nan's Venn diagram circle activity.

Nan was the only PST who did not use technology during the study. But like Beth, all of her lesson plans involved the use of hands-on activities and she incorporated ongoing assessment into them, as well. Moreover, she did not feel as though she got enough evidence from the clinical interview. In fact, she felt that since the assessment tool emphasized angles at the beginning, her student may have not discussed other properties of shapes, thinking the focus was only on angles. She developed a lesson plan to address her assumption. She explained that she:

> will do a shape sort with my student (see Figure 14). I'm doing this not necessarily to teach her the hierarchy of quadrilaterals, but to see if she realizes that shapes have characteristics (other than having angles). If/when she catches on to identifying the characteristics of shapes, we may *explore* the hierarchy of quadrilaterals based on the characteristics she has identified… I will make sure her titles match the rules she is using to sort the shapes as accurately as possible. After she has noted major characteristics: number of sides, side lengths, and angles… (Lesson Plan 1)

When discussing the purpose of the lesson, Nan stated that it "is to get the student to recognize the existence of characteristics and to then begin to compare characteristics of shapes" (Journal 3). In relation to the LTs, her plan to get her "student to recognize the existence of [properties]" of shapes, is within the LT level she assigned her student. She placed her student between a 1.2 and a 2.1, which means that her student could correctly identify shapes as a whole, however, incorrectly described their properties. The second part of her objective was to have her student begin to compare the [properties] of shapes, which skipped approximately 3 levels of geometric thinking.

To get her student there, Nan had the student leading the exploration of shapes, compare and contrast shape characteristics based on what the student understood, and create sets of rules. The student would be actively engaged in all of these cases, while both the student and Nan continued to assess her understanding of shapes.

In her second lesson plan, Nan continued to do a Venn diagram shape sort with the student. This time, she focused solely on quadrilaterals. However, she wanted the student to examine the shapes even further by using a ruler to validate her visual interpretations. For example, if her student said that only one set of sides are equal in length, Nan had the student use a ruler to defend her claim. This is similar to what Beth planned to focus on while using the software program when having the student compare the "visual" side lengths to the "numerical" side lengths. In Figures 9 - 13, both the side length measurements and the angle measurements are listed at the top, along with a picture of the shape. All three students were continually asked to use the resources around them to explain and prove their thinking.

Learning by doing is a broad term. In this study it is defined as not only having the student doing something, but also thinking about what they are doing. The lesson plan activities of all three PSTs had the potential to actively engage the students in the learning process and discover new concepts. Their lessons called for active engagement by manipulating shapes on a computer screen, using cut-out two-dimensional shapes to be placed inside a Venn diagram, and other hands-on activities, all the while asking the students to compare and contrast the new information to the previous knowledge. The researcher acknowledges that these activities took place in a one-on-one setting and is not implying that they would translate similarly in a small-group or whole group environment. Nevertheless, these opportunities for the students to discover concepts were within their grasp because each PST used the LTs to identify the next steps needed to take with the students.

Research Question 1 Sub-Question: Instruction

- In what ways do PSTs use LTs to instruct lessons on a geometry topic?

Learning Trajectories and Instruction

The National Council of Teachers of Mathematics (2014) state that, "Effective mathematics teaching elicits evidence of students' current mathematical understanding and uses it as the basis for making instructional decisions" (p. 53). Furthermore, the evidence elicited should be grounded in how students' thinking evolves over time. They recommend LTs as a source for identifying markers of student thinking (NCTM, 2014). Finally, using purposeful tasks that elicit students' thinking and making instructional decisions based on their thinking is often referred to as formative assessment.

Through the use of lessons that encouraged active learning, PSTs used LTs to develop tasks to elicit their students' thinking and adapted their instruction based on their students' thinking. Therefore, the PSTs used LTs as a formative assessment tool during instruction.

Theme: LTs as a formative assessment tool through active learning. The theme for instruction contains three sub-themes: 1) how PSTs elicited their students' thinking during instruction; 2) how PSTs used evidence of their students' thinking to modify instruction; and, 3) how PSTs observed and evaluated the results those modifications had on their students' thinking. Presentation of the data for the three subthemes will not be compartmentalized. Instead, they will be presented in a cyclical fashion for each PST.

This study illustrated the result the modification had on the student's LT level of geometric thinking for that particular situation. Moving forward, it is important to note that a change in the level of geometric thinking for a particular task does not necessarily imply that the students' overall level of geometric thinking is now situated in the new level. Battista (2012) suggests that students can think at different levels (both higher and lower) for different tasks depending on their familiarity, or unfamiliarity, with the content.

Beth's instruction. Beth went into lesson 1 with the purpose of gathering further evidence of her student's knowledge of the properties of shapes and to create a [cognitive] map that she and her student could refer back to. In order to make the "[cognitive] map of her student's thoughts on various shapes," she created an anchor chart (see Figure 8) so he could tell her "all of the things he knows or notices about the various [quadrilaterals]" (Lesson Plan 1). Here is what transpired in Lesson 1:

Beth: Okay, what do you think about squares? What makes a square a square?

Student: They have four sides, four points, and four 90 degree angles.

Beth: Anything else?

Student: They could be rectangles, but rectangles can't be squares.

Beth: Okay. So we'll just put that next to the square. That's our starting point. What about rectangles, what do you know about those?

Student: They have two pairs of parallel lines.

Beth: Okay. Anything else?

Student: 90 degree angles.

Beth: K.

Student: Can't be squares.

There was an opportunity to dive into higher-level questioning when her student said that squares "could be rectangles," but rectangles "can't be squares." She could have asked him to justify his reasoning, moving the conversation into a discussion of the hierarchy of quadrilaterals. Instead, she chose to record it on the sticky-note and save that questioning for later when they explored the shapes on the computer. In this next exchange, Beth acknowledged his imprecise name for a rhombus. But more importantly, she provided evidence of modifying instruction based off of the information she elicited.

Beth: Now when I met with you last time you said this [rhombus] is the same thing as a diamond. So I might say rhombus, but you can say diamond if you prefer that. And what do you know about rhombuses?

Student: They're like squares, I guess.

Beth: Why are they like squares?

Student: 'Cause squares, if you turn 'em like- See how that one's right there? Turn it like that, and it's like a diamond.

Beth: Mmm, okay. So if I were to take that one and turn it so that it's like standing up on this angle it would be like a rhombus?

This evidence suggests that the student is not thinking at a 2.3, and instead, looking at some properties as visual wholes, rather than focusing on the properties of the shapes, which is approximately a level 1.2. He also made no reference to the measurements of the properties, something Beth had planned to do in the Lesson Plan 1. They continue exploring the rhombus:

Okay. Do you notice anything else about 'em?

Student: I see acute angle, obtuse, acute, and obtuse again.

Beth: Alright, and anything else?

Student: It’s a little bit- it kind of looks like a parallelogram.

Beth: It does. Doesn't it? You're giving us lots to explore on the computer

After her student told her that the rhombus “kind of looks like a parallelogram,” she told him that he has given them lots to explore on the computer. This is evidence that she was considering modifying her lesson based on the student’s thinking she elicited. While she already had plans for exploring the shapes on the computer, the data gathered seemed to be giving her ideas of other things she might explore, specifically, parallelograms and rhombuses. Although it never came to fruition, she did confirm in Interview 2 that she was planning to go that route, later in the lesson:

> I want to make sure and see if he truly understands why a square is a special type of rectangle. If he's not completely sold on that then I want to work on that some more If he is, then I either want to go on to the rhombus or the parallelogram. The rhombus because it's still focusing on the side lengths. The parallelogram because I saw in the assessment that he thought that that was a rectangle and I just kind of, I'm curious to see if he would notice that now, since we stated the properties of rectangles. (Interview 2)

This highlights her attention to his thinking in conjunction with the direction of the lesson, which is indicative of using formative assessment strategies.

Beth continued to elicit his thinking and actively engaged him in the learning process by having him explore the different quadrilaterals using Geometer's Sketchpad, starting with the square. "As he drags the shape around, I will ask him what he notices" (Lesson Plan 1). Her reasoning for this was driven by the LTs. For example, in Journal 3, Beth explained that by, "exploring individual shapes and stating the observed properties, the student will be progressing toward a level 2.3 in which he will be able to 'formally name and describe shapes completely and correctly.'"

What is interesting about this activity is its dual purpose. Beth was continuing to elicit the student's thinking, while simultaneously providing him a chance to discover new ideas through the dynamic software. The anchor chart had static pictures of shapes, which limited him to recall all of the properties of the shapes from memory. But by using Geometer's Sketchpad's *Measured Quadrilateral Makers* activity (see Figure 15), he could use his recollection of the properties of shapes and compare that to the geometric constructions of the shapes as he manipulated them. Moreover, the hot links – which are discussed in the planning sections - embedded in the software tool gave him the opportunity to focus on measurements of the shape properties.

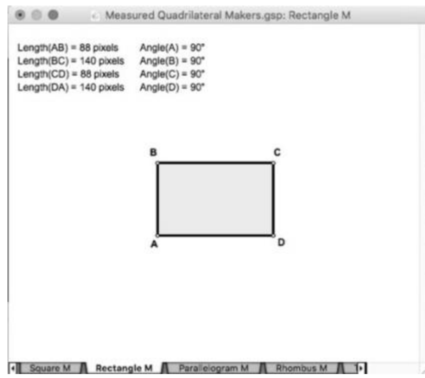


Figure 15. Image of the Geometer's Sketchpad's *Measured Quadrilateral Makers* activity.

After spending time dragging and exploring the different quadrilaterals, Beth had her student return to the rectangle maker and asked him what shape he saw. The proceeding discussion followed:

Student: That's a rectangle.

Beth: It's a rectangle? How come?

Student: All of them are the same [pointing at the angles].

Beth: Can you make a square out of that rectangle?

Student: Nope.

His final response here, along with his statement earlier in the lesson that "squares could be rectangles, but rectangles can't be squares" (Beth's student, Lesson 1), prompted Beth to modify her instruction. Her intent going into this part of the lesson was to focus on measurement using the software tool to help him discover the importance of the size of lengths and angles when identifying properties of shapes (Lesson Plan 1). However, this would not have happened regardless of her initial intent, because earlier in the lesson she made a decision to change her focus to parallelograms since she got "lots to explore" from the student when working with the anchor chart. This prompted her to potentially modify her instruction, yet it never came to fruition. Instead, in response to her student saying "nope," an exploration of the hierarchy between squares and rectangles became the new direction, or modification, of her instruction based on the information she elicited. Earlier in the lesson, it was mentioned that Beth had a similar opportunity to modify her instruction and move into a discussion of the hierarchy of shapes. However, she chose to record it on a sticky-note and hold off. And even though this concept is roughly four levels above where Beth placed him, it is still along the trajectory for geometric shapes, which means LTs continue to drive her instruction while making instructional decision through formative assessment.

Beth followed his lead and continued to probe his thinking and had him explore further. She had him drag the vertices of the rectangle maker to see if he was able to make all four sides with equal length, which he did. Then, while referring back to the anchor chart she reminded him of his previous statements while he examined the shape he made in Figure 16:

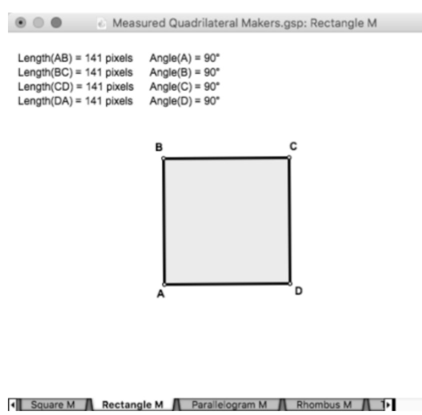


Figure 16. Rectangle with four congruent sides.

Beth: You said that we had to have four sides, four points, which we have. You said that we had to have four 90 degree angles, which it looks like we have that, right? And then you just said that all four sides have to be the same length, right? So do we have that here?

Student: Mm-hmm.

Beth: We do? Oh. So this is showing us all the properties that a rectangle has to have?

Student: Yes.

While looking at the square he had just made, Beth asks:

Beth: So if I pointed at that and I said "Hey, what shape is that?" What would you tell me?

Student: A shape.

Beth: Yes, but what name? What would you tell me?

Student: A square or a rectangle.

This is a challenging, yet enlightening, result from the modification Beth made to her instruction. During the clinical interview, and in Lesson 1, the student claimed definitively that squares could be rectangles, but rectangles can't be squares. With this logic, if it were true, the software program would work much differently than it did. For example, if squares “could be” rectangles, then the logic that would follow is that the user could make rectangles while using the square maker. Conversely, the user would not be able to make a square with the rectangle maker if rectangles “can’t be” squares. Both of these scenarios are incorrect, which is why Beth chose to modify her “instruction in hopes of having him discover this misconception” (Journal 4). The rectangle maker retains the geometric construction of a rectangle. Meaning that no matter how you manipulate it, you will always be looking at a rectangle. A rectangle that happens to have all four sides the same length is also known as a square.

Beth modified her instruction and continued to actively engage him in the learning process. Instead of telling him what she wanted him to know, Beth had her student drag the rectangle’s sides until all of them were equal. In that moment, the student was looking at a rectangle that had opposite sides congruent. It just happened to be the case that all sides were congruent, but it still implied that opposite sides were congruent, as well. Her student was looking at a rectangle, but he was also looking at a square. Both names are accurate. And her student came to that realization when he said he was looking at "a square and a rectangle."

While it is only a discussion about two shapes, the level of geometric thinking involved is one shy of doing geometric proofs. She followed this up by asking another string of questions to elicit his understanding even further:

Beth: Alright, so we said that the opposite sides are equal in length. So why then is a square a rectangle?

Student: All sides are equal in length.

Beth: Mm-hmm.

Beth: But does it also have- Are these two the same? What about these two? [pointing at opposite sides of the rectangle].

Student: Yeah.

Beth: So could I say then that these [pointing at opposite sides of the square] - That this also has opposite sides that are the same length? I mean we know that they're all the same length, but is it okay for me to say that?

Student: Yeah.

Beth: And that it has the two pairs of parallel lines?

Student: Mm-hmm.

Beth: And that it has the four 90 degree angles?

Student: Mm-hmm.

Beth: So it sounds like you just kinda proved to me why a square is a special type of rectangle. What about when we look at a rectangle. And I remember you told me that a rectangle can't be a square. When we look at what we say a square has to have, what's the rectangle missing?

Student: Well all the sides being equal.

Beth: Yes. Excellent.

After asking him why a square is a rectangle, her student responded only with “All sides are equal in length.” And although having all sides equal in length implies that opposite sides are equal, Beth spent time probing him further. Even though he responded mostly with affirmation to her questions, he validated his understanding after her final question: “When we look at what we say a square has to have, what’s the rectangle missing?” He stated that the rectangle does not have to have all sides equal, yet he agreed that they both have opposite sides equal, which revealed that he was beginning to see the interrelatedness of the properties of these two shapes; a level 3 concept according to Battista (2012).

Beth elicited and modified her instruction with results that stretched farther along the trajectories than her initial intent. She went into the lesson planning on “getting a “running list” of what the student sees in the various quadrilaterals we will be exploring” (Lesson Plan 1). Her goal was not specific to proving that a square is a special type of a rectangle. Yet, through eliciting her student’s thinking and modifying her lesson, she arrived at what she felt was a significant point in the lesson. In Interview 2, when asked to identify an important moment in the lesson, she stated that:

> I'd say probably at the end when he was able to prove using properties why a square is a special type of rectangle and why it can't go the other way. That's really kind of dependent on how much he still gets it next time and if we have to kind of like move backwards a

little bit again for him to truly understand it, or if he really got it in that moment. (Interview 2)

In the last sentence of this quote, she alluded that more eliciting would need to take place to find out if her student "truly understands" that a square is a special type of rectangle. And if not, she was willing to modify her future instruction based on his thinking.

Melissa's instruction. Like Beth, Melissa used the Geometer's Sketchpad program to "encourage discovery-based learning" throughout all of her lessons (Journal 4), and even started out her first lesson using the same Quadrilateral Maker as Beth (See Figure 15). In Lesson Plan 1, Melissa chose to "focus on identification of shapes, properties, and justification." And wanted him to look for things he noticed as he dragged the shapes. The following dialogue recounts her student actively engaged in the software program, while Melissa used probing questions to elicit his understanding of properties of squares, rectangles, and parallelograms.

She has him start with manipulating and exploring the square with the square maker activity to get him used to the software and then moved to the rectangle maker in Figure 2:

Melissa: All right. So let's do it with a rectangle now. You can play with that however you want and tell me what you notice about it.

Student: Well, I already see, it goes like a pattern 88 to a 103; 88 to 140, but the angle always stays the same.

Melissa: What do you think those numbers and those numbers are for?

Student: The size, like the length they are apart.

Melissa: Oh, so like these lines along there?

Student: Mm-hmm.

Melissa: And what do you think these are, because these say 90 degrees, and I don't think any of those are 90. You can play with it. You can move it around so that way you can see if the numbers change or what they do.

At this point, Melissa is focused on eliciting his understanding of the properties of rectangles and has not yet modified her instruction. She, however, drew his attention to the four measures of 90 degrees above the shape (see Figure 17). As mentioned in the planning section, prior to this lesson, Melissa's student was only focusing on side lengths during the clinical interview, which led Melissa to see if she could draw his attention to angles, as well. He identified them as the measure of the angles in the rectangle and proceeded to drag the shape to explore its properties further. Melissa asked if the measurements on the screen were changing as he dragged it:

Student: Yeah, but the angle doesn't, but the lengths do, sometimes it changes around.

Melissa: Okay, so only the length changes on those again?

Student: Mm-hmm. Every time you move the rectangle, it's always having a pattern like 129, 205, 129, 205 (Figure 17).

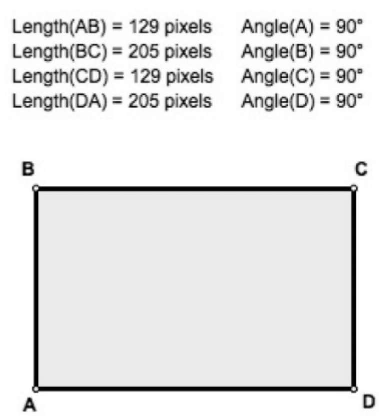


Figure 17. Image of the rectangle made by the student using Geometer's Sketchpad.

Here, her student was beginning to recognize angles as a property of rectangles and that those measurements do not change, which, again, was the goal of her lesson to explore and elicit his thinking to get him to recognize other properties of shapes besides just the side lengths (Lesson Plan 1). However, just before moving on to the parallelogram maker, Melissa made a modification to the lesson that had an impact on her student's thinking.

Before discussing that, it is necessary to go back and identify a major misconception both her student, and others at his level, often have. Students at a level 1.1 incorrectly identify shapes as visual wholes. Changes in a shape's orientation from what the shape is presented as "traditionally," impacts their ability to identify the name of a particular shape (Battista, 2012). For instance, when looking at a right triangle with one of its vertices as its "base" (see Figure 18), her student thought it sort of looked like "a triangle," but ultimately decided it was not (Clinical Interview 1). What happens next provides evidence that the student is beginning to move beyond a focus on shape orientation and into the properties of shapes when identifying them.

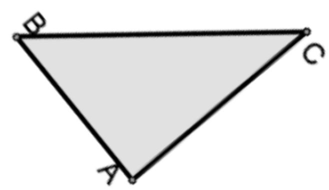


Figure 18. Image of a right triangle "on its vertex" created by Geometer's Sketchpad.

As previously mentioned, Melissa's student noticed two patterns about the rectangle: that no matter how he dragged it 1) the angles stayed at 90 degrees; and, 2) opposite sides always stayed equal. Hoping to capitalize on his recognition of these patterns, she decided, in that moment, to not move on to the parallelogram maker, and, instead, modified her instruction in hopes of helping the student clear up his misconception about shape orientation. She wanted to "find out if he would focus on the shape as a whole, or if he would use the patterns to name the shape", so she "created a slanted rectangle" to find out if he would still name it a rectangle, which it is (Interview 2). If so, he is beginning to focus on properties of rectangles. If not, then the visual-whole of what the shape typically looks like still persist in his thinking (Interview 2).

Melissa took over control of the computer for a moment and began dragging one of the corners of the rectangle in the rectangle maker (see Figure 19).

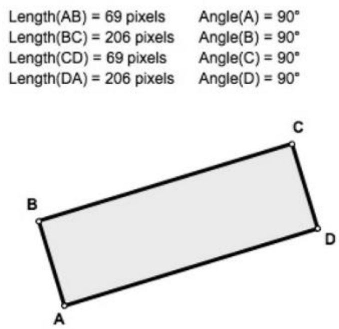


Figure 19. Image of a "slanted" rectangle created with the Geometer's Sketchpad.

She then asked:

Melissa: What do you notice about that?

Student: It's going sideways, but it still stays the same, but the number on the length changes [she is not able to spin it without the side lengths moving up or down a little], but the angle stays the same.

Melissa: What would you call that shape?

Student: A square, but no a square. Ah, I don't really know what this is?

Student: [After looking that the shape and the measurements above] Oh, it's still a rectangle, it's just that it's really crooked!

Even though she manipulated the shape right in front of him and only turned it counter-clockwise about 45 degrees, the student's first reaction was to call it something other than a rectangle. After he had time to process what he was looking at, he came to the conclusion that it was, indeed, a rectangle. First of all, this illustrates the challenge students at his level have with moving away from only recognizing shapes as a visual-whole to focusing on their properties for identification. Secondly, by using prior knowledge of her student's thinking, coupled with the new evidence she elicited, Melissa was able to modify her lesson in order to alter his misconceptions about the orientation of the shape, getting him to focus on its properties, as well.

Melissa continued eliciting his understanding of shape properties through an exploration with parallelograms (see Figure 20). This time, she continued capitalizing on his recognition of patterns and began exploring similarities and differences between the properties of squares, rectangles, and parallelograms. The discussion went as follows:

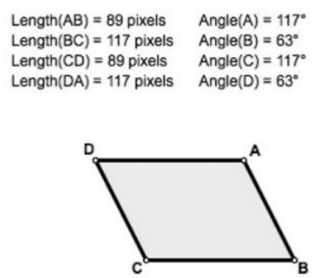


Figure 20. The parallelogram maker activity from Geometer's Sketchpad.

Melissa: So this one, do you notice anything?

Student: Now they're both just a pattern. Just the angles, I think they move. Do they move around?

Melissa: Does this one move around?

Student: Like do the angles change?

Melissa: Let's find out.

Student: Yeah, so now the angles and it like changes now, but back on a rectangle and square, it didn't change the angle, but now the angle's coming into the pattern just like the rectangle did on the length, and now it's doing the length and then angle as a pattern. (Lesson 1)

Melissa changed the focus of her lesson in order to follow her student's lead, allowing him to investigate his own question about the angles. He discovered there was not only a pattern with opposite angles, but that the opposite sides revealed a pattern, too. With that, he was able to make connections to the patterns of properties he identified earlier with squares and rectangles. Melissa's modifications to the lessons moved the instruction beyond her original focus of doing "an activity that has the student identifying properties of shapes" (Lesson Plan 1), to not only focusing on using properties of shapes to name them, which prevents misconceptions when the orientations are not "typical," but comparing and contrasting properties of multiple shapes using the patterns the student recognized.

This shows how she used the information she elicited to modify an instructional decision later on when she chose to change the orientation of the rectangle, which resulted in the student beginning to understand that the properties of the shapes matter, not its orientation.

Melissa created tasks that actively engaged her student. Her use of the Geometer's Sketchpad software allowed the student to manipulate the shapes and make discoveries about properties of shapes, under her guidance. While her tasks were open-ended, the probing questions and modifications to her lessons closely followed the LTs, which were apparent throughout the aforementioned examples.

Nan's instruction. Unlike the other two PSTs, Nan did not use technology during her lessons. Instead, she focused on using concrete materials, like polygons and Venn diagrams. She entered the lesson with a focus on eliciting her student's thinking about properties of shapes. No significant modifications were made in the first lesson. Instead, Nan spent the entire time eliciting her student's thinking. In Lesson Plan 1, she explained that she would:

…do a shape sort with my student (see Figure 14). I'm doing this not necessarily to teach her the hierarchy of quadrilaterals, but to see if she realizes that shapes have characteristics (other than having angles). (Lesson Plan 1)

Nan had the student lead the exploration of shapes. The student was actively engaged while Nan elicited her thinking. For example, after the student determined the placement of different shapes in the Venn diagram, Nan asked her to identify things they all have in common. The student responded with, "They're shapes," so she probed her further until she recognized that all the quadrilaterals have four angles.

Nan: So right now you have shapes and they all have four angles. So right now, according to what you told me, I could put this [rectangle] here [with the squares]. Could that one [rectangle] go there [with the squares]?

Student: It could. I mean it's ... it really could go there because it has four angles just like a square.

Nan: Mm-hmm.

Student: And it has four sides.

When Nan asked her why she did not include the rectangle with the squares and whether or not it could go there, the student began focusing on similar properties of both shapes and believed that it could since it had four sides and four angles, which began moving her thinking beyond visual recognition of shapes and into a focus on their properties. However, when asked what was different about squares and rectangles, she reverted back to seeing the rectangle as a visual-whole when she said things like the rectangle is "long and it is wider" and "bigger" than a square.

This was a small, yet significant, moment in the lesson, nevertheless. Nan's goal was to assess her ability to recognize other characteristics besides angles, which she did. Eliciting her understanding remained the main priority throughout most of the lesson. And although it was not her focus, she did get the student to discuss some of the similarities in the properties that squares and rectangles have. Recognizing that the two shapes share common properties like four angles and four sides is a step toward higher-level thinking, which Nan capitalized on in the next lesson.

In her second lesson, Nan continued to do a Venn diagram shape sort with the student. This time she focused solely on quadrilaterals, with the intent of asking the student "specifically about shapes next because she was able to recognize it generally, but I want to see if she can actually start naming shapes and identifying shapes based on those properties, not just noticing them overall" (Interview 2).

The student was given sheets of paper and asked to label them with categories. Nan told her to "do whatever you want" when the student asked how Nan wanted her to label them. However, Nan did add one rule: all the quadrilaterals must go on the paper. The student paused for a moment to think.

Student: I don't know. 'Cause anything on the paper has to be a quadrilateral.

Nan: Mm-hmm.

Student: I know that the trapezoid is a quadrilateral. But, can we do a quadrilateral can be a square or a rectangle or something?

Nan: But, then, all the trapezoids are going to be alone. 'Cause they're all a family. Trapezoids and kites are both. Like if they had a name, their last name would be quadrilateral. Because they're all quadrilaterals, but the trapezoid, oh, these are the special quadrilaterals that that's a certain person. But, they're also a quadrilateral, like my name's [Nan Smith]. I am a [Smith], but I'm also Nan. If we only put on our quadrilateral page squares and rectangles, we're not including their brothers and sisters.

Student: Can I do quadrilateral shapes on one paper and then, can I do trapezoid shapes on another paper and then, kites on another paper?

Nan: Okay. But, aren't trapezoids quadrilaterals, too?

Student: I was not thinking about that.

Nan: Are they or are they not?

Student: They are!

In this exchange, her student picked squares and rectangles as the categories, which would have left the trapezoids and kites out of the diagram. Nan stopped and took a moment to clarify why she could not do that in that situation. She tried making the words trapezoid and quadrilaterals analogous to the first and last name of a person. Her student seemed to understand this well enough that Nan modified her instruction to incorporate an exploration of the logical inferences that can be made about quadrilaterals, which will be illustrated next. A similar example to this would be a rectangle and parallelogram. The logical inference would be that if the definition of a parallelogram is two sets of parallel lines, then a rectangle is a parallelogram since it always has two sets of parallel lines.

Using the evidence of her student's thinking, Nan modified her lesson by setting aside the Venn diagram and, instead, with the student, built an impromptu hierarchy of quadrilaterals similar to Figure 21.

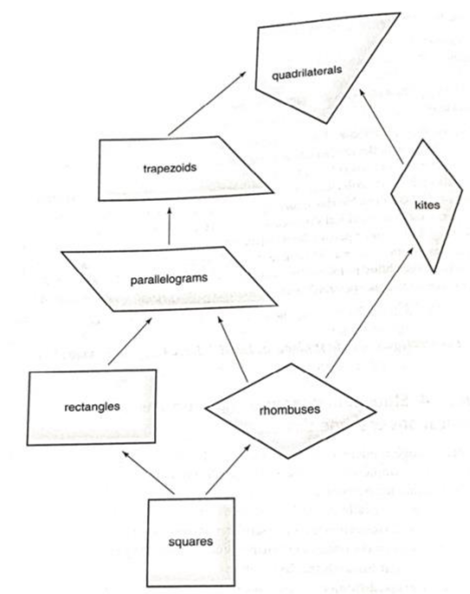


Figure 21. Image of the hierarchy of quadrilaterals.

Nan started with the quadrilateral and connected it to the trapezoid with a line, while her student identified the similarities and differences between both sets of properties. This continued all the way down to the square. Near the end of the lesson, she handed the student two different size squares from the pile of quadrilaterals and the following exchange took place:

Nan: Okay. Now, let's look at this one and this one [two squares]. Are they quadrilaterals?

Student: Yes.

Nan: Are they trapezoids?

Student: Yes.

Nan: Are they parallelograms?

Student: Yes.

Nan: Are they rectangles?

Student: Yes.

Nan: Are they kites?

Student: Yes.

Nan: Are they rhombi?

Student: Yes. Wait. Yes!

Nan: Are all sides the same length?

Student: Yes.

Nan: Do they have two pair of adjacent sides that are the same length?

Student: Yes.

Nan: Do they have four sides?

Student: Yes.

Nan: Do they have at least one pair of parallel sides?

Student: Yes.

Nan: Do they have exactly two pair of parallel sides?

Student: Yes.

Nan: Do they have four right angles?

Student: Yes.

Nan: Then, what are they?

Student: Everything!

Nan: They're everything?

Student: Yes!

Nan: And, they're special because they are?

Student: Oh. Squares!

This exchange is the culmination of the modification Nan made to the lesson. At first glance, it might be easy to dismiss this with the majority of the questions only needing a “yes” response. However, when Nan asked her student “what are they?”, and she responded with, “Everything,” a lot was revealed about what she discovered. One thing she discovered was that not only do squares have two pairs of parallel lines, they also have at least one pair, which is essential to understanding the interrelatedness between properties of shapes and higher-level geometric thinking. When the student looked at a square, she also recognized that it is a rectangle, rhombus, parallelogram, and all the other quadrilaterals. It is "everything."

Through the use of concrete materials and modified instruction, Nan was able to actively engage her student at a level 3.4 by getting her to connect the square to all of the shapes, which is six levels higher than her initial assessment of 2.1. Rather than compartmentalizing each level and only teaching to mastery in one before moving on to the next, Nan elicited her student's thinking and modified her instruction so that she could work with higher-levels of geometric thinking.

The activities of all three PSTs actively engaged their students in the learning process while PSTs made adaptations to stay within their zone of proximal development. In order to do this, PSTs had to collect data about their students and determine whether or not the students were assimilating or accommodating their own understanding of geometric shapes. This could be seen as a cyclical process of assimilation or accommodation of new schemas of the PSTs understanding of LTs and tutoring and also their understanding of their students. This required them to focus on, and interpret, the same assimilation or accommodation processes as the students themselves went through as they made sense of the work done with the PSTs.

PSTs' students were actively engaged by manipulating shapes on a computer screen, using two-dimensional shapes and organizing them in a Venn diagram, along with a variety of other examples. In addition, the students were asked to compare and contrast new information to previous knowledge. The PSTs used LTs to develop the tasks that elicited their students' thinking and made modifications to their instruction to adapt to the students' thinking, which is indicative of formative assessment strategies. All of which was supported by constructivist learning theory.

Research Question 2: In What Ways do PSTs Reflect on Their Use of LTs and Plan to Use LTs in Their Future Instruction?

To elicit data for research question 2, the PSTs were asked to reflect on their use of LTs throughout interviews and journal reflections. In addition, they were asked to consider how they might use LTs to guide their future teaching. These two categories are presented separately. However, in some cases, their reflection on their use of the LTs morphs into ideas and beliefs that could be argued as a discussion of future use of LTs. The new schemas revealed by the PSTs was supported by constructivist learning theory. In order to avoid confusion, this section includes those gray areas, recognizing they represent, in some ways, a new schema of teaching and learning as a result of their lived experience. In the next section, the data on future use is specific to times when PSTs were asked to discuss future use, explicitly.

Research question 2a: in what ways to PSTs reflect on their use of LTs. A major theme came out of the PSTs' reflections on their use of LTs: PSTs felt the LTs helped them create and use a *cognitive road map* of their student's thinking. This cognitive road map mirrors the teaching cycle in that assessment drives planning, which in turn drives instruction, and the cycle starts over again. This is not a new concept to the field of education. But the role LTs played in the PSTs' experience through this cycle was unique. While the cognitive road map is the overarching theme, subthemes for assessment, planning, and instruction, immerged.

All three PSTs reported having a positive experience using LTs to guide their teaching (Interview 3 and Focus Group Interview). When asked to reflect on her overall experience with LTs, Beth felt it opened her eyes to the complexities of teaching:

> I feel like before working with these, even though I know that it's way more complex than this, you have your students in the middle, and then you have your higher level students and your lower level, and so it was just those three categories, even though I know that that's not it, but now with the learning trajectories, I can see the different levels, and it's much more complex than just higher, [average], and lower levels. (Interview 3)

Beth added, "the first word that comes to mind is differentiation' (Interview 3), which Melissa also discussed in Journal 5:

> We learn a lot about differentiated instruction and we read about different theories that people have but we don't apply them while we are in school…With learning trajectories we are able to design lesson plans with every student being thought about and that is spectacular in itself. I feel like using [LTs] with the right guidance will change mathematics instruction in the future. (Journal 5)

Melissa also claimed that using LTs to differentiate instruction might not only help with students "falling behind or in the average place," but they can be used with gifted students, as well.

Nan also saw the LTs as a tool for differentiation and added to this idea. She described LTs as a "vehicle for moving up [students'] conceptual understanding and critical thinking (Journal 5 and Interview 3). Not only did she see LTs as a tool for differentiating instruction, she also surmised that LTs might aid in *equitable learning*, taking teaching and learning beyond *equal learning* opportunities. She clarified this by stating that:

> Equal learning opportunities would be me as the teacher giving all students the same lesson. That's equal, right, but that is not what's going to help all of the students grow the same amount. If I want all of them to grow the same amount, so an equal amount of growing, it means that I interact with them differently, making it equitable rather than equal. That's

> where the learning trajectories become super-important, because this is ensuring that this is an equitable lesson, not an equal lesson. (Interview 3)

She takes this idea another step further by connecting it to the debate in education about assessing growth vs proficiency. She said, "The equitable is going to be teaching and assessing growth. The equal is going to be assessing their [thinking] and comparing it to the standard, to the average" and that "the equal is a mistake… because that's not giving all students the same opportunity to grow, which is ironic because it's supposed to be equal. In reality, it's not" (Interview 3).

Nan's view on growth versus proficiency was brought up in the Focus Group Interview. Melissa added:

> I remember discussing the battle between growth and proficiency in another course. If a student isn't proficient in what they're doing, they're not going to have that growth or any ways, from where they are. I think that they go hand in hand, and we need to pay attention to both and not just so much the proficiency anymore. State level testing is a lot about proficiency. If we gave the same test to every student, they're not all going to be proficient, but it may show growth from where they were before. I feel like that makes a big difference, rather than them all being on the same level because that's just unrealistic. You're putting children in a box at that point. (Focus Group Interview)

Beth agreed and felt that LTs give teachers "something concrete to show this is where they were and this is where they've gone. This is how they've grown. Without the learning trajectories, you don't really have anything to base that off of," to which the other two PSTs agreed (Focus Group Interview).

Cognitive road map. Learning trajectories hypothesize levels of sophistication the mind goes through over time as it wrestles with mathematical ideas. Each level on the trajectory tells how a child at that level understands the mathematical concept and gives examples of what that might look like. All three PSTs opined that using LTs aided them in finding out where the child is, where they need to go, and how they can take them there; likening this to the creation of cognitive road maps of their students' thinking to guide their teaching (Focus Group Interview).

In Journal 2, Melissa stated that "So far I am enjoying using the [LT levels] because it helps me to see where the student is, how we could review, and where they have potential to be in the future under the correct instruction." She expanded on this idea later in the study when stating that LTs help teachers:

> …to know where to go and it's kind of a map, and the map that works for every student which is really cool about it because it adapts by itself, and you have to be the one to say, "I feel like they're at this level, and we can move forward," or, "I feel there's still some challenge left for that person," you don't stop with a child, and I think that's the coolest part about it. (Interview 3)

There is an important distinction to make here. Melissa is not just seeing the LTs as a way to map student thinking, but also to let student thinking determine movement along the LT map. There is a sense of a symbiotic relationship between the two.

Beth also revealed evidence of thinking of cognitive road maps. Interview 1 took place after the initial assessment, but prior to the first lesson. During the interview, she said:

> I'd say they're a way of seeing where a student's at and developmentally what the next steps are or what the goal is for the next steps. For where you want them to go next and you can see what's too far, what they're not ready for and what they've already mastered and you're really able to figure out what you want to do and what you need to do to get them to the next place. (Interview 1)

Like Melissa, this idea stayed with Beth through Interview 3 and even the Focus Group Interview, which Nan agreed with even back as far as the first interview. She noted that she found "them extremely helpful in lesson planning and assessing students because if you don't know where they are how do you know where to take them" (Interview 1). For all three PSTs, and on multiple occasions and from multiple data, continually pressed on how they liked using LTs because it allowed them to create a cognitive road map of their student's thinking: where they are (assessment), where they need to go (planning), and how they are going to get them there (instruction). This study narrows its focus one last time to discuss, explicitly, the data revealed for assessment, planning, and instruction.

Assessment: where they are? Beth "really liked using the LTs as an assessment tool because it was the first step in creating the road map" of her student's thinking. However, as previously mentioned in the assessment section of research question 1, Beth did feel that "determining his level has proven to be quite challenging" (Journal 2). Although Beth struggled with determining a level early on in the study, she later found that struggle to be rewarding. In Journal 5, which took place after the final lesson, Beth talked about this reward in effort:

> One thing I really like about the CBA is that it didn't give you the level the student was at. That was for me to determine based on the answers provided by the student. This caused me to really dissect and think about the responses I received. I couldn't just take the number of questions the student got right or wrong and match it to [a] level. (Journal 5)

Melissa, like Beth, found assigning a level to be challenging, but for a different reason. Melissa was worried about the potential negative implications of *labeling* a child. This is brought up in Journal 1 and Interviews 1 and 2 as she felt early on that leveling her student is equivalent to labeling her student and she did not want to put him "into a box." Nevertheless, she chose a level for her student and, later, found quite the opposite effect.

Melissa found she liked using LTs for assessment because she was able to keep finding out where the student was to see if she was challenging enough (Goldilocks syndrome – not too high, not too low), staying within her student's ZPD through ongoing assessment. She liked being able to tell if the instruction was "too much or not enough, and…continue challenging the student, but not so much that it was over their head to where they just felt overwhelmed and nervous" (Interview 3). She continued to discuss this during the Focus Group Interview, showing a complete contradiction to one of her early beliefs:

> The cool thing about [LTs], too, is that, even though there are levels, it doesn't put the child in the box of just that level. They can be in between two levels. They can be on their way to one level, or maybe they need to be stronger in a different level. They maybe got skipped over or something like that. That's the cool part about this. They're very adaptable. They can be used for any child at any time. If there aren't levels, you can make them for them. I really like that about it. (Focus Group Interview)

Although both Beth and Melissa changed some of their early beliefs and challenges, this did not mean they did not reflect on other issues for LT use with assessment. In particular, both were unsure on how to use them on a larger scale. Beth (Journal 5) felt that the clinical interviews were an important part, but failed to see how she could use it as often as she would like and did not feel she could use them "one-on-one with everyone" (Interview 3). Similar, Melissa did mention in Journal 2, that she was "wondering is how you could do this with every child in a classroom and still get this much out of it."

Nan did not find the assessment portion of the study challenging like the other PSTs. Nan's reflection on her use of LTs for assessment got her thinking about not only how she could use the results to guide her planning, but also how they could be used, later, as a summative assessment that has been adapted by both the LTs and the students' thinking:

> I think that if you start with a pre-assessment (1) you can assess where they are on the learning trajectory, right, which then prepares you to get the student ready for the next step and whatever the trajectory would say. Beyond that, when you get to the final assessment then you know how to create the assessment in a way that the student is going to best understand it so if they're at a lower level you're not going to give them an assessment that says ... you might give them an assessment that says that this angle on this quadrilateral is 90 degrees instead of just visually leaving it blank and making them assume. Because then they say, "Yes, it is 90 degrees and I know 90 degrees is an important part of this quadrilateral." (Interview 1)

As an assessment tool, Nan is thinking about LTs as a formative assessment to adapt instruction. She also sees LTs as a tool to drive the creation of the summative assessment; this makes summative assessment a much more fluid concept.

Planning: where they need to go? In Journal 2, Melissa stated that the LTs "had a large impact on the lesson plan." She discussed the following:

> Once you find the student's level you can then develop a plan of action for where you want to take this student. One of the best part of it is that you can develop a plan to work with each individual's needs. Some students will need extra time with certain concepts and activities. With learning trajectories that is now acceptable because each student has a plan

> and goals of their own. Without this, students that are below average will be getting left behind and students that are above average will be bored. (Journal 2)

By the end of the study, Melissa claimed her idea of planning changed:

> Using learning trajectories changed my thinking about lesson planning all together. Using [LTs] opened my eyes to a large problem, the fact that we are not differentiating planning…We commonly use differentiation in our program after writing full lesson plans and then treating students that aren't at the "average" level like an afterthought. With learning trajectories we are able to adapt our lessons as we plan instead of after. This means that we would have well developed ideas of what students are going to be doing at each level and how you can push every student to the next level. (Journal 5)

Melissa felt, beforehand, that teachers decide on lessons and plan for them, but that the idea of differentiation gets put in at the end. Whether or not this is true for most teachers is not the point The point is, this how Melissa has interpreted what she has been taught about and explored with lesson planning up to this point in her tenure as a student. That interpretation now sees LTs as a way to drive the lesson and activities from its inception.

In Journal 5, Beth stated that "learning trajectories definitely had the most influence on the assessment and planning portion" and, that "learning trajectories were really useful in that, because that's what told me where my student was and where they needed to go, and what they needed to accomplish to get there" (Interview 3). However, she did note that she "didn't really have to refer back to them for the next lessons, and I don't know how much of that is because I already kind of had an idea, or if they just are more useful as you start each level" (Interview 3).

Nan, like the other two PSTs, found "them extremely helpful in lesson planning and assessing students because if you don't know where they are how do you know where to take them?" and that later, when outlining her planning process, Nan said she would have "the LTs next to her while planning so that you can see what comes before the level and after the level of the topic you are covering" (Interview 2). And, finally, she talked about how she "matched her [student] up with her level on a trajectory and then considered for my planning, okay, how do I move her to the next level, using the content that I have, the content being the vehicle, but the next level on the trajectory being the destination" (Interview 3).

The data on planning showed that the PSTs used the LTs to guide their instruction, which was not surprising, but rather, expected. At the very least, this section confirms that not only did they use the LTs to plan their instruction, but, more importantly, the LTs continued to be the driving force behind the idea of the cognitive road map. The last section discusses instruction.

Instruction: how they are going to get them there? In research question 1, the data revealed formative assessment as the theme for instruction. In this section, the PSTs' reflection on their use of LTs during instruction revealed that what is generally called formative assessment, they called *live questions*.

During Interview 3, and in Journal 5, Beth made it clear that she felt that the LTs did not specifically have an impact on her instruction. She felt that, indirectly, through assessment, and planning, they did, but not while instruction was actually taking place. Nan, on the other hand, did, in fact feel that LTs impacted her instruction, directly. She coined the term "live questions" for the group.

In Journal 5, Nan stated that LTs "are useful during instruction because they prompt a teacher to ask live questions that continue to help the students understand the topic more deeply and therefore increase understanding." She was asked to describe her interpretation of what it means to ask live questions. She stated that they are:

> …reactive questions. You have your questions that you planned beforehand, but sometimes a kid will do something and you need to be able to react to what they're saying in a way that helps them move forward, and doesn't just give them the answer. (Interview 3)

When describing it to the other PSTs during the Focus Group Interview, Nan stated that "because I was focused on the trajectory and leveling up, my questions were different." And:

> What I ended up calling that was live questions, so what you think of on the spot. As your student does something, [they are] your reactive questions rather than your planned questions. You need those to help a kid move. You can't really plan for every single thing that might happen in the lesson. The trajectories, I think, really help with creating those live questions or helping you be prepared for those live questions because you have that goal in the back of your mind. (Focus Group Interview)

Melissa agreed with Nan and added:

> Actually, I think the learning trajectories helped me to bounce not only up levels, but back. If a student recognized something, but he was just missing one piece, I could bounce back and ask a question that I knew he knew the answer to. That would lead him to find the next thing. (Focus Group Interview)

Lastly, although Beth initially felt that LTs did not play a role during her instruction, she changed her mind at the end of the study. In Journal 5, she stated:

> As far as teaching, the learning trajectories helped me know where to go based on responses I received from the student. If I didn't receive the response I was looking for I used my knowledge on what the student should be able to demonstrate to guide what I did next (whether I was successful or not). It wasn't really something I had to think about. I know

that the more I work with learning trajectories, the more I will be able to use the knowledge I have gained and be able to adapt my teaching accordingly. (Journal 5)

In their reflection on their use of LTs, the PSTs felt that they can be used as a tool for differentiating their instruction. In particular, 1) they saw them as a way to create cognitive maps of their student's thinking; 2) through assessment, the PSTs found a starting point on the map; 3) through planning, the LTs gave them another point on the map that they thought they could get their students to; and, finally, 4) during instruction, the LTs guided the student questioning similar to the way a navigation device (GPS) would reroute your directions if you made a wrong turn.

These reflections are supported by constructivist learning theory, zone of proximal development, and scaffolding, which guide this study. Preservice teachers provided evidence both in the actions and their reflections on their experiences that schemas held prior to the tutoring project were changed into new schemas, which were profound to them. For example, they believed that LTs can be used to stay within students' zone of proximal development. And, LTs can also be used a tool to provide scaffolding to their students as they bridge the gap to the next level of the trajectories.

Research Question 2b: In what ways to PSTs plan to use LTs to guide their future instruction?

Preservice teachers were asked to identify ways in which LTs might guide their future instruction. This happened during the final stages of study. Specifically, during Journal 5, Interview 3, and the Focus Group Interview. Since the entire study was done with one-on-one instruction, the questions meant to elicit their potential future use targeted around how they might use LTs to support small and whole group instruction. In this last section, evidence from the data is provided to highlight two themes that emerged: 1) Hetero-Homogeneous Grouping; and 2) Building Up - Whole Group Instruction.

Theme 1: hetero-homogeneous grouping. All three PSTs consider LTs as a tool for grouping their students for instructional purposes (Focus Group Interview). Moreover, they did not always want homogeneous groups. And though they discussed groupings that consist of students with varying levels of sophistication, they did not want the ranges of the members of the group to be too far apart. The PSTs saw this method of grouping as *Hetero-Homogeneous*, in that students' levels are not too similar, but also not too far apart, keeping with knowledgeable others who are within their zone of proximal development. The following addresses this idea presented by each PST.

In Journal 5, Beth discussed how she might use LTs for small-group instruction. She stated that she "can most definitely see [LTs] being a large part of small-group instruction, especially in the initial grouping of students for small-group work." In order to obtain this initial grouping, "pre-assessments would have to be done individually," similar to what she did for the tutoring project, but does not, up to this point, see any other way of establishing the levels of all the students in an entire class. While her point may seem obvious, this coincides with her uncertainty with how LTs might be used with whole-group instruction, which will be discussed later. Nevertheless, how she plans to group her students is of note.

During Interview 3, Beth mentions not only that she would use the information gathered to help determine how she would group her students for small-group instruction, but that the LTs showed her that there are different ways in which to group students. She was then asked to go deeper into what she meant by this:

> I see myself grouping them ... I don't know, it's hard to explain. [takes a short pause to regroup her thoughts] Where one student is struggling, another student is strong, but still kind of at around the same level and those groups aren't permanent. They're going to move around as they develop and even just to move around just to be working with new people and seeing new ideas and stuff like that, I guess. (Interview 3)

She goes on to describe her small-groups as homogenous in that “if you had to label them, the students would all share the same LT level” (Interview 3). However, since there is variance within each level, she would strategically put students together so that one of them is a little stronger within that same number compared to the others, so there is a kind of heterogeneous sense to each group, as well. She thinks of her future groupings as more hetero-homogeneous, rather than just one or the other.

Like Beth, Melissa also discussed using the students’ levels to establish hetero-homogeneous groups. Initially, though, she describes her grouping as homogeneous:

> I see using group instruction not to separate but to embrace each student’s knowledge. Once students are assessed they will be able to be seen in groups with students that are at their level. I see this being similar to reading or writing when the students get called over to a table with the teacher to improve their skills (Journal 5).

She goes on to discuss how her grouping method might assist students who are in-between levels:

> Several students will be in-between levels and this will be a good way to address it. For example a student that is between a 1.2 and a 2.1 should first meet with the group of 1.2 students. When they meet with this group they will get a good review to build strength in their knowledge. They will be able to contribute to the group to help others learn and progress to the next level as well. When that student leaves the 1.2 group to advance to 2.1

> they will be able to learn a lot from the students that are then going between level 2.1 and 2.2. They will constantly be progressing because of their classmates and when they need guidance I will be able to provide it (Journal 5).

Once Melissa considers students who are in-between levels, her focus shifts from students working in groups at the same level, to a more fluid sense in which a student moves between groups, similar to stations in a classroom. She discussed this further during Interview 3:

> When you have a student that's on a certain level, probably few other students in your classroom are going to be on that level, so then you can take students that are all at that level and pull them over to you and create activities that work for them because they're all at the same level, and maybe some students in that group are moving onto an upper level, or maybe they just came from a different level, so they're both contributing things that maybe the upper ones forgot, or maybe the lower ones don't know yet. And so they're learning from each other. (Journal 5)

Like Beth, Melissa recognizes the potential variances in students' levels of sophistication that she will most likely have to deal with once she has her own classroom. Rather than viewing this as an obstacle, they both envision it as an opportunity for their students to grow with each other. Her reason is that it would not only help involve everyone, but that it would be a way to build the confidence of the student who is the "highest" in that group. Then, just like Beth, her groups would continually change, she would adapt it so that the student who was once the highest in one group, is in the middle, or lower, in order to challenge them. Melissa believes that using this form of grouping will build strength and confidence in a student's ability by thriving and sharing in groups where they are the highest level in that group. And then, by moving them to a group where they are a lower level, they can learn from peers who are knowledgeable others.

Nan shared similar ideas to both Beth and Melissa. Nan believes that LTs will be “an important tool in determining individual and group work” (Journal 5) for her future classroom. She went on to say:

> …after assessing students’ levels, an instructor may group students in homogenous groups to give them specific activities to advance them to the next level of understanding together, or they make heterogeneous groups so that different level students can share their processes and thinking, so that all levels are exposed to different manners of understanding a topic (Journal 5).

Like Melissa, Nan initially described her strategy for creating heterogeneous and homogeneous groups as mutually exclusive. She felt that there are “advantages to both, depending on the topic and the lesson” (Interview 3). She added, “I really think most of the time - and I don't have experience, so I don't know - but my guess is I would make heterogeneous groups more than I would make homogeneous groups” (Interview 3). She goes on to describe the process she might use to establish heterogeneous groups and that it:

> …doesn't necessarily mean I have one of each level in a group. That might mean that I have my high 2s ... like 2.2s, 2.3s ... together, and then my 1.1s and 1.2s together. No, I wouldn't even do that. I would do like 1.1 to 2.1, and then 2.2 and up, and maybe mix it around a little bit sometimes. I would never have exactly the same group, to make sure that every level is getting a little taste of the other levels, so that they can see how other kids are thinking about things. (Interview 3)

She came to an interesting conclusion here. She initially described putting groups together where the students are only one level different from each other. She changed it so that there was a greater range, yet she did not talk about having the lowest level student working with the highest level student in a group. In her view, she seemed to be thinking of the levels as two categories in which to place the students. She did, however, mentioned that she would make sure that students are "getting a little taste of other levels" (Interview 3). Like the other two PSTs, Nan sees establishing groups that are closer in range with their levels of understanding. Moreover, she, like the others, made it clear that students are not locked in to a certain group. Instead, movement in and out of groups are important to them.

All three PSTs gave compelling evidence that they plan to use LTs in the future for small group instruction. These groups will be made purposefully using results from the "cognitive mapping" they reflected upon in the last section. They talked about how they don't want to do homogeneous groupings if they don't have to. However, the heterogeneous groups that they do create will involve groups of students whose levels are much closer to each other at times, making them sort of a hetero-homogenous group. For example, one talked about putting 1.1s and 1.2s together at one point, and then another time putting a different range together like 1.2s and 2.2s, but close enough so that all could contribute to the learning, be leaders at time, and be the ones learning from others.

Heterogeneous and homogeneous groupings should not be a new concept to the PSTs, so it is not surprising that the two are discussed in conjunction with small-groups. What was unique was that all three described setting up groups that were within the students' zone of proximal development (ZPD). This is similar to the work they did during the one-on-one tutoring where they chose to work within the ZPD of the students' ability. They took that experience and kept both the individual's ZPD in mind while imagining a classroom where the small-groups would also have a ZPD element to them. This idea is intriguingly present in their use of LTs during whole-group instruction, as well.

Theme 2: building up. Finally, the PSTs were asked how they might use LTs during whole-group instructions. Beth was unsure of how she might use them in this way, but was open to the ideas of the others when it came time for the Focus Group Interview. Ultimately, all three PSTs agreed that LTs could be used with whole-group instruction in a way that they believed would make the student feel safe and build confidence through successful interactions with the entire class. The following data reveals a strategy of whole-group instruction in which a teacher, through the knowledge of their students' LT levels, might explore the topic and ask questions in a way that increases along the trajectories in order to give all students a chance to share ideas within their level of understanding, giving each the opportunity to both contribute to the group and two hear the ideas of other students above, below, and at their level of understanding for a particular topic.

Prior to the Focus Group Interview, Beth was unsure of how she might use LTs for whole-group instruction. "It's somewhat hard to determine what role learning trajectories might play in whole-class instruction … I don't know how I would use the learning trajectories in a way that would influence whole-class instruction" (Journal 5). Her sentiment is expressed further during the third interview:

> Whole group, I think, would be the most difficult because I have a hard time seeing how I would ... I see how I would introduce how I would cover the topic as a whole group, I guess. It would just have to be like an introduction and then I imagine them breaking off into smaller groups to work on different things, but all the same thing. The same thing, but with a different focus. (Journal 5)

At this point in the study, Beth is confident with how she might use LTs during individual and small-group instruction. Ideas of their use in whole-group instruction were eluding her. Although she could not imagine their use herself, she did express a curiosity of what the other PSTs might come up with, leaving her open to the possibility (Journal 5 and Interview 3).

Melissa was able to imagine the use of LTs during whole-group instruction. She saw them as a way to push all of her students at once:

> I would incorporate every level in the classroom. Starting with the lowest level and working my way up. Students that are at the lower level will get the instruction that they need plus see a little bit of what is to come. Students that are in the middle will get some review and will see what will be soon in the future. Students that are at the highest level will get review to build strength in their understanding and will progress further to finding out what comes next (Journal 5).

She goes on to give an example of what this might look like:

> Something like the shape shifter program [from the Geometer's Sketchpad program] that we used for my lesson with my student would be great to use for any student. Students just developing an understanding for shapes will get to play with them a little bit. Eventually they will move into noticing properties that make shapes what they are. Once they have developed that they can look for patterns and start to add proper names to what they have already discovered. Once they know the names they can start to incorporate lines of symmetry, finding angle measures, using equations, etc. See how fast things can escalate! (Journal 5).

Melissa seems to believe that the very nature of the tasks she creates for whole group instruction can have a positive impact on the entire group, regardless of level of understanding, as long as she considers all levels of her students when creating them. In her previous example, she described an activity where the whole-group is using a computer program at the same time. It is understandable if the reader thinks this is an obvious way to work with a whole-group in a way that allows students to be successful. Especially since the student would more than likely be using the program individually or with a partner, which does not fit the description of whole-group instruction, but more like individual or small-group. However, during the third interview, she clarifies this idea and how it might be used with or without technology through the types of questions she would ask of the entire group.

In Interview 3, Melissa described a setting where she would have different levels of questions to ask during whole-group instruction. She would adapt her questions to fit the students by going forwards and backwards along the LTs. And at one point she talks about how she would ask lower level questions so those "students can be successful and then increase the complexity so everyone contributes" (Interview 3). This is the beginning of the theme of *Building Up* for whole-group instruction. Nan describes a similar approach.

Like Melissa, Nan imagined creating a range of questions based on her students' LT levels. She went a step further than Melissa by giving details about how she would use them. She explained that during a lesson she would "walk around the room and see how the kids are doing their work individually and mark down what the specific kids that are at different levels" are doing (Interview 3). She would then:

> …ask the kids at the lowest level to first talk about their idea, and then find the kid that's at the just-next level, ask them to expand on that person's idea, and then ask the next level to expand on that person's idea. (Interview 3)

This is an interesting way to imagine whole-group instruction. One might think that this would single out lower-level students, but Nan felt that it would not be obvious to the class since the students would be randomly placed throughout the classroom (Interview 3). Furthermore, she felt that for the students they would:

> Not necessarily even coming up and sharing your own idea except for the first person, but showing that, "Oh, I understand how [the first person] thought of it, and here's what I added on just a little bit," and the next person comes up. It helps that idea grow, and doesn't make it seem like, "Oh, that person's a lot higher than everybody else." It was a group effort for everyone to think of what the main idea is and growing it into whatever the whole concept is. (Interview 3)

During the Focus Group Interview, Melissa and Nan shared their ideas about how they would use LTs during whole-group instruction. Not only did Beth consider their ideas as valid ways of using them during whole-group instruction, but she also confirmed that she would like to use them in the same way (Focus Group Interview).

Melissa shared some of her own struggles with figuring out their use for whole-group instruction:

> At first, I struggled with the question. Then, when I thought about it, it won't hurt students to see what is to come. With doing whole group instruction, you're doing that. You're still continuing to have them aim higher always, while teaching all of the other students at the same time. (Focus Group Interview)

Melissa really seems to embrace a positive view on using LTs during whole-group instruction, giving a sense their use would contribute to the success of all students, regardless of their level of understanding. Later on during the Focus Group Interview, Nan gave a descriptive account of how she might use them. This led into a discussion between all three PSTs which revealed a collective idea of how they might use them and what impact they could have on their teaching. The following details this account, starting with Nan:

> If you're doing the whole class, and let's say that everyone has their own worksheet, and you could even be doing it in groups, too, I guess, but you walk around while they're working on it and see where each person is at in their level. Note some things down and different methods that are being used around the room … So you walk around and start noticing trends…and noting who is doing which trend. Then, identify those trends in relation to what level they would be correlated with. Call up a kid to share their ideas from the lowest level first, and build your way up. (Focus Group Interview)

Nan then goes on to explain the next steps she would take. She points out that she would call upon students in a systematic way to starting with the lowest level of understanding and moving along the trajectory for that topic (Focus Group Interview). She reiterates the fact that the order in which the students are called upon is not obvious to the students. Her point is valid since most students fall on different levels of a trajectory depending on the topic, so the students might not recognize a pattern. Nan goes on to describe the hypothetical interaction:

> You're not actually asking them to put up their own idea, but it's adding on to someone else's. You keep doing that with all of the levels to build-up to the full concept or the deepest concept, so that everyone feels like they have a piece in it and no one feels like they're stupider than anyone else because it was a class collaboration to come up with this whole idea. It's not leaving one kid behind saying, "Oh, you thought less than this kid thought." (Focus Group Interview)

Melissa added:

> Right. I think that doing that makes every child feel valuable. That they're building on things. They don't even need to know what level they are. You just need to go "I really like how you did that. Can you show the class? I think we can build from that." Then that student, the next one, comes up, and they're building on it. They don't know their level because now they're not placing themselves in that box, but they know that they meant something to that class. That's the coolest part about that thing.

Beth included:

> I like that, too, and it makes sense. My student, he was at a level 2.2, but he still had some of the knowledge that he needed for a 2.3. He was missing this puzzle piece. I feel like doing that would help fill in those gaps that students have in different places.

Melissa agreed with Beth and Nan added one last point, which everyone agreed with:

> You can do it, too, where if ideas or methods are really different, you would call kids up and say, "How does your method compare to this one? Where do you see similarities between your two methods? Where can you notice that the same things are happening, just in different ways in your two different methods?" That's a really good way, I think, to compare a kid who's at a lower level to a higher level and help them take those next steps, if they're struggling to get to the next step. Seeing the relationship between their ideas and higher level's ideas will help them understand it a little bit better and be able to move on. (Focus Group Interview)

This discussion illustrates the PSTs desire to work with their students within their ZPD, while simultaneously not excluding anyone from participating during whole-group instruction. Melissa felt that "…something like [this] makes the classroom more of a safe place to learn" and that

“You're giving everyone a fair opportunity to learn at their level and move from [there]” (Focus Group Interview).

This final theme is related to the three PSTs’ views on how LTs might be used during whole group instruction. Essentially, they see the classroom broken up into different levels, in general, based on whatever topic they are covering. For example, imagine having a low, medium, and high group for shapes. All three want all of their students to be able to participate, but also feel safe and successful. So what they imagine is adapting the questions they ask to move along the trajectories from low to high. This is supported by concept of scaffolding instruction. Rather than asking one question and giving everyone a chance to answer it, they imagine asking a question that they feel is within range of what their lower students can answer and give them realistic opportunities at answering it and feeling successful. Then, they would move to another question that is more advanced, which the medium level students can answer. In that moment, the lower level students have felt success and, might feel more confident, and safer, and want to either try to answer the question too, or, at least they would try to make connections to it in the moment because they feel success and are willing to challenge themselves.

Summary of Findings

Chapter 4 presented the themes and patterns developed from an analysis of the data and were supported by constructivist learning theory, zone of proximal development, and scaffolding. The major findings of the PSTs’ use of LTs during the study were that they:

- showed flexibility in identifying their students’ level of thinking during assessment;
- planned lessons within their students’ zone of proximal development and created lessons that encouraged active learning; and,
- used LTs as a formative assessment tool during instruction.

When asked to reflect on their experience throughout the study, PSTs felt the LTs helped them create and use a *cognitive road map* of their student's thinking. And when asked to consider ways they might use LTs in the future, they considered them as a tool for creating *hetero-homogeneous* groups for small-group instruction and for *building up* during whole-group instruction.

CHAPTER 5: DISCUSSION

Introduction

Chapter 5 summarizes the findings of this study and evaluates and connects it to the research questions. As discussed in chapter 4, the findings of this study are supported by constructivist learning theory, zone of proximal development, and scaffolding. A discussion of the results will include a comparison to the literature review from Chapter 2. The limitations of this study are addressed, as well. Additionally, not only will this chapter discuss the implications this study has on the field of mathematics education, but also suggest recommendations for future research.

Discussion of Findings

This study addressed the following questions:

1) In what ways do PSTs use LTs to assess, plan, and instruct lessons on a geometry topic?
2) In what ways do PSTs reflect on their use of LTs and plan to use LTs to guide their future instruction?

In the next section, the data that emerged from each research questions will be reviewed and interpreted.

Research question 1 sub-question: LTs and assessment. According to the data, PSTs showed flexibility in identifying their students' level of geometric thinking about shapes during assessment. The PSTs were not given specific instructions on how to use the data they collected from the clinical interview. Instead, they were asked to use the assessment tool with their student and to determine for themselves what level they believed their student to be along the LTs for shapes. And while all three eventually chose levels for their students, they also concluded that it was not necessary to label their student at only one level. However, Beth and Melissa took a unique path to that conclusion.

Theme: flexible level identification. Beth and Melissa were hesitant, early on, to level their students. Melissa went as far as to not want to predict what level her student might be when asked to in Journal 1, prior to the clinical interview. She was concerned that any preconceived notions of her student might negatively impact her interview with him (Journal 1). Later, during Interview 1, Melissa expressed her concerns about being fair to her student and that she did not want to put a label on him.

It seems as though Melissa was initially connecting the idea of choosing a "level" for her student with the concept of "labeling" in education. Labeling students often carries a negative connotation, so her hesitation is warranted. A below-average label, some would argue, may follow a student from grade to grade and condemn them to inadequate teaching practices that perpetuate the student's plight. Melissa was genuinely concerned that if she chose a level for her student after assessing him, it would constrain her ability to teach him moving forward.

Melissa's interpretation of leveling changed after the clinical interview. She became comfortable with assigning a level to her student once she decided that the levels were "very adaptable" (Interview 1) and that it was not necessary to assign a student to one level if they are showing signs of understanding in multiple levels.

Beth did not struggle with choosing a level in the same way as Melissa. Beth found it challenging to choose a single level when she was looking over the data she collected during the clinical interview. Her student showed signs of understanding at various levels, making it difficult for Beth to pinpoint her student's highest level of proficiency.

Both Melissa and Beth used a similar version of the phrase "putting them in a box" when discussing their hesitation to choose a level for their students. They did so, however, for distinct reasons. Melissa felt that the act of choosing a level for a student would put them inside a box, depriving them the opportunity to learn things outside of the metaphorical box. More specifically, she felt that if a level is given to a student, then that student will only be taught to that level and would not get to expand their thinking to higher levels. Beth used the phrase for a different reason. She felt her student's understanding of geometric shapes was sort of scattered all over the place. And therefore, choosing level was like trying to fit him into a box. A box in which he would not fit into nicely a neatly. Both views of leveling were intriguing.

Their initial reluctance to choose a level for their students eventually led them to surmise that they were not required to choose only one level. All the negative restrictions and limitations that leveling might cause were their own preconceived notions. However, these preconceived notions were changed once they had the chance to reflect on their student's understanding of shapes. And while Nan did not have initial concerns about leveling, she ultimately came to the same conclusion about not needing to assign only one level to her student.

Research question 1 sub-question: LTs and planning. All three PSTs used the results from the assessment data gathered in the clinical interview and through formative assessment during instruction when determining learning goals. As mentioned in the sub-question on assessment, the data revealed that the PSTs showed flexibility in identifying students' level of geometric thinking, meaning they did not feel it necessary to identify their students at only one level. They showed similar flexibility when planning lessons. Two themes emerged from the data on LTs and planning: PSTs planned lessons that both stayed within the students' zone of proximal development and used activities that encouraged active learning.

Theme 1: planning within students' zone of proximal development. Melissa and Nan both created lessons that stayed within their students' ZPD by planning to work with concepts no higher than two levels above their level of proficiency. Beth, on the other hand, felt she could only work one level above her student's level of proficiency. This was not due to a lack of belief in her student's ability or willingness to work multiple levels up, but rather an issue with the necessity of accurate use of formal language at the higher levels. In the earlier levels, informal language is expected. As students evolve, it is expected that both informal and formal language are used interchangeably. One student might use informal language to describe parallel lines, but use formal language when discussing congruency of lines and angles. Another student might do the exact opposite. Regardless, both students would be considered to be proficient at the same level. This flexibility may explain why Nan and Melissa were able to work multiple levels up. Beth did not have this flexibility, so her reach for higher levels were more than likely stunted by the necessity of formal use of language.

It seems as though it may have been too challenging for Beth and her student to work two levels up. Beth's student was proficient at 2.1, which meant that he could informally describe parts and properties of shapes. One example might be when describing properties of a square he would say that all four corners look like "Ls", but would not identify them as right angles or 90 degree angles. Students at a level 2.2 blend their descriptions of properties of shape with a combination of formal and informal language. For example, when describing a rectangle, he would state that opposite sides are equal (formal) and has four corners that look like an "L" (informal). He could work within a 2.2 with some quadrilaterals, but not all. In order to work at a 2.3, Beth would have been trying to get him to formally describe shapes completely and accurately. While 2.1 and 2.2 seem to go together, it appears that working within level 2.3 requires a strong foundation in the previous two. That isn't to say that she could not have taken one of the quadrilaterals and focused on just that one and do lessons at a 2.3, working on identifying all of its formal descriptions. Instead, Beth chose to focus on getting her student at a 2.2 with all of the quadrilaterals. This makes sense because the ultimate goal, beyond what the PSTs would have time to do with their students, is to get students to understand the hierarchy of quadrilaterals. It is possible that she thought about the quadrilaterals as a group and wanted to move her student's understanding of that group simultaneously. It should be noted that Beth's thinking evolves later in the study. She eventually decides to move away from a focus on all shapes and works with her student at a higher level with just squares, rectangles, and rhombuses.

Theme 2: lesson plan activities that encourage active learning. As mentioned in chapter 4, the activities used by all three PSTs had the potential to actively engaged the students in the learning process to discover new concepts. Moreover, PSTs included activities that encouraged students to discover concepts that were within their ZPD through active learning.

Melissa used The Geometer's Sketchpad exclusively throughout all three of her lessons. The Geometer's sketchpad software program allowed her student to explore draggable geometric shapes while maintaining their defining properties. While manipulating the shapes, her student was able to validate, contradict, or develop new understanding of the defining characteristics of different quadrilaterals. Not only was her student able to actively engage in the discovery of the defining properties of the shapes, but he was also able to assess his understanding through the same program. For example, the "Predict and Check" activity discussed in chapter 4 encouraged this active learning. Her student had to look at a static picture of a quadrilateral and make predictions about how it might be manipulated into other quadrilaterals. Knowing that the shape must always retain its fundamental properties, he was able to instantly assess his knowledge of each shape.

Like Melissa, Beth also used the software program to encourage active learning throughout all of her lessons. However, she also used a poster as a visual display of his understanding that he could keep referring to as his understanding evolved. The poster changed in conjunction with his conclusions. This was a creative way to use technology alongside a concrete manipulative.

Nan did not use any technology, at all. Rather, she focused on hands-on activity using Venn diagrams and paper cutouts of different shapes. Nan had her student lead the exploration, comparing and contrasting shape characteristics based on what the student understood, and create sets of rules. The student was actively engaged in the learning process while both the student and Nan continued to assess her understanding of shapes.

All three PSTs created lessons that encouraged active engagement through the either manipulating shapes on a computer screen, using cut-out two-dimensional shapes with a Venn diagram, and/or other hands-on activities. These opportunities for the students to discover new concepts were within their grasp because the PST used the LTs to identify the next steps needed to take with the students during planning, which is not surprising given that the PSTs were working one-on-one with a student

Research question 1 sub-question: LTs and instruction. PSTs used LTs as a formative assessment tool during instruction. This happened through the use of lessons that encouraged active learning. PSTs used LTs to develop tasks to elicit their students' thinking and adapted their instruction based on their thinking. The following theme emerged.

Theme: LTs as a formative assessment tool through active learning. The theme for instruction contained three sub-themes:

1. PSTs elicited their students' thinking through the use of purposeful tasks and discussions that were grounded in the LT levels assigned to each student;

2. PSTs used evidence of their students' thinking to modify instruction which were elicited by the task or through probing questions; and, finally,
3. PSTs observed and evaluated the results those modifications had on their students' thinking.

During instruction PSTs used LTs as a tool for formative assessment, which allowed them to work within the students' ZPD. An interesting observation about the data was that all three PSTs decided they would teach by going up the levels, instead of going back. That is, they created lessons that started at, and then built upon, the students' level of understanding. This is also not surprising given that the PSTs were working one-on-one with a student, but this may have looked quite different in another context.

Suppose the PSTs did not have access to LTs and, instead, were given a fifth-grade curriculum and asked to teach a lesson on geometric shapes. The data from this study reveal that the lessons from the fifth-grade curriculum would probably have been way beyond the students' level of proficiency. During instruction, there is a good chance the PSTs would have found both their students, and themselves, frustrated with the content. And after doing some formative assessment, they might have realized they had to "back-up" the lesson, which can be hard, psychologically, on both the students and the PSTs. Instead, by starting with what the students were able to do, the PSTs worked within their ZPD and worked their way up the levels. One example of this came from Beth.

Beth identified her student at a 2.1, but noted that he had elements of 2.2 and 2.3. Some of her activities addressed these, but she also worked all the way up to a 3.4 (hierarchy of shapes). Early on, her student said, to some degree, that a square is a rectangle, but a rectangle is not a square. Beth tried to build off of that and get him to talk about why believed this to be true. She wanted him to prove it. At the same time, she moved over to the rhombus to see if he could use the same logic when comparing a rhombus and a square. This was all done even though he was much lower with all the other shapes. In a sense, Beth was bouncing back-and-forth within a broader ZPD.

There are a number of ways she could have approached this. One option might have been to gradually, and meticulously, work within one level, only to move on once a student has shown proficiency within that level with all the quadrilaterals. As mentioned in the planning section, Beth did this early on in the study, but she eventually moved away from it. She recognized that her student was at different LT levels for different shapes, which is not uncommon. Battista (2012) acknowledges that students can think at different levels in different situations. For example, if one has only been exposed to a couple of quadrilaterals, it seems only logical that a person would be at a higher level for those shapes and at a lower level for the ones they are unaware of, or with which they have had little experience. Beth recognized this in her student and tried to capitalize on it by using his strengths to build up his weaknesses, which is something that both Nan and Melissa did, as well.

The study concludes that the overarching theme for research question 1 is “flexibility”. Within and across all cases and all sub-questions, LTs helped PSTs to be flexible throughout the teaching cycle. Starting within their students’ ZPD, PSTs used LTs to elicit their students’ understanding and modify their instruction in the moment. This allowed them to work within a broader, or more flexible, ZPD. Since there was not a control group in this study, the research can only speculate as to the significance this may have had on the students’ understanding. Nevertheless, the PSTs found the LTs useful during instruction and felt they made progress with their student throughout the study. The overarching theme for the first research question being that the PSTs not only found flexibility in their use of LTs, but also flexibility with assessment, planning, and instruction, because of LTs.

Research question 2 sub-question: reflection on use of LTs. PSTs were asked to reflect on their experience throughout the entire study. The goal was to gather data that could paint a picture of what impact LTs had on their understanding of teaching and learning. Differentiation was the common theme from all three. Again, during Interview 3, Beth stated that:

> The first word that probably comes to mind is differentiation. I feel like before working with these, even though I know that it's way more complex than this, I guess I thought of students as like...You have your students in the middle, and then you have your higher level students and your lower level, and so it was just those three categories, even though I know that that's not it. But now with the learning trajectories, I can see the different levels, and it's much more complex than just higher level, [average], and lower level.

Differentiation typically implies the idea of modifying instruction to meet the needs of individual students. It is not surprising that this would be at the forefront of the PSTs' minds as they worked one-on-one with a student, tailoring their instruction to that student. Nor is it surprising that they connected differentiation ideas to assessment, planning, and instruction, since they were continually asked to connect their experiences with all three throughout the study. Interestingly, though, the PSTs likened the use of LTs as a way to create and use cognitive road maps of their student's thinking. Specifically, these cognitive road maps helped them to see: where they are (assessment), where they need to go (planning), and how they are going to get them there (instruction).

Research question 2 sub-question: plans for future use. PSTs were asked to envision how LTs might guide their future instruction. Consistent with the data from their reflection on their use, PSTs felt that using LTs in their future classroom will be imperative to successful teaching and learning. Since the study was done with one-on-one instruction, the questions were designed to elicit how they might use LTs during small and whole group instruction. Two themes emerged from the data: 1) Hetero-Homogeneous Grouping; and 2) Building Up - Whole Group Instruction.

Theme 1: hetero-homogeneous grouping. One theme derived from the data involved grouping students for small-group instruction. All three PSTs considered LTs as a tool for grouping their students based on their levels of proficiency. However, they specified that their goal would not be to set up homogeneous groups of students with approximately the same level of geometric thinking. Instead, they felt that it would be better to create groupings that consist of students with varying degrees of levels so they can work with and learn from *knowledgeable others*. This is referred to as heterogeneous grouping. This style of ability grouping was discussed often during their coursework, so it was not surprising that they focused on it. Interestingly, they did not feel it best to leave the groups open to random selection of its members. Rather, the students in the group would have varying levels of proficiency that are not too similar, but also not too far apart, keeping with knowledgeable others who are within their zone of proximal development.

One issue with heterogeneous grouping is when two students are at different levels of proficiency, only one of them can be the “knowledgeable other” who supports the other’s learning. This does not imply that a student with a higher level cannot learn from a student with a lower level of understanding. In fact, being the knowledgeable other gives the student the chance to increase their depth of understanding.

The elegance of the PSTs’ style of grouping is that students are never put into a box (a reference to an issue the PSTs had with leveling early in the study). In any lesson, a student is either learning something new or deepening their understanding of a concept. Constant rearrangement of groupings would allow the majority of students to play different roles in different groups. Simply put, during some activities, a student is learning from knowledgeable others, while other times that same student is the knowledgeable other.

It is inevitable that with a given mathematical concept one student will have the highest level of proficiency and another will have the lowest level of proficiency when compared to the entire class. In that case, the highest level student would always be the knowledgeable other and the lowest would always be learning from the knowledgeable other. This does not mean that they will always be in that role. As Battista (2012) noted, levels of understanding can fluctuate. For example, Beth's student had a higher level of understanding of the hierarchical relationship between squares and rectangles. But this level of understanding did not extend to other quadrilaterals. It is possible that he could be the at the highest level in class when everyone is working on squares and rectangles, but another student might be above him when working with parallelograms and rectangles. The same could apply to the lower level. It is important to keep in mind that nothing is fixed. Like the PSTs discussed, the LTs are flexible in their use.

Theme 2: building up. Lastly, PSTs considered how LTs could be used during whole-group instruction. All three felt that, if used in a systematic way, LTs might help students feel safe and build their confidence during whole-group discussions and activities. Using LTs as a lens for instruction, the teacher could create activities and ask questions that increase in sophistication in order to give all students a chance to share ideas, giving each the opportunity to both contribute to the group and to hear the ideas of other students above, below, and at their level of understanding.

This is an intriguing idea. That is, any question asked or activity done during whole-group instruction has varying degrees of impact on each students' learning based on, in big part, the students' level of sophistication with the content. By using LTs to develop questions, or by knowing the LTs well enough beforehand, teachers can ask purposeful questions that target specific students who fall within a certain ZPD.

An argument against such an approach might be that there is not enough time in the day to individualize instruction for all students. Imagining a classroom of twenty-five students with twenty-five individual needs, the thought of differentiating for all appears daunting, at best. Though it need not be seen that way. Instead, imagine the same twenty-five students as one entity. One learning group. Just like an individual student has a ZPD, so does that one group. The group has gaps in its understanding, just like an individual would. By using LTs as a lens for teaching and learning, the teacher can place an individual student somewhere along the path of a trajectory and build them up from there. The same can be done with the entire group. Just like the PSTs in this study realized that their students didn't fit completely into one level of a trajectory, the whole group does not, nor would it, fit into one level as well. By doing this, the teacher could possibly avoid teaching only to the middle, and they might also avoid teaching outside of the ZPD of the entire group.

Teachers should not overemphasize any single aspect of teaching. Instead, there should be an even balance of attention given to assessment, planning, and instruction (API). More importantly, API should not be thought of as mutually exclusive, nor should they be only thought of strictly in a cyclical order. Instead, all three are intertwined, like a pretzel. Each one influencing, and being influenced by, the other two, equally.

In conclusion, each of the PSTs seemed to have been enlightened by the experience. In thinking ahead, Beth stated:

> I can't imagine teaching effectively without using learning trajectories. They offer an immense amount of insight into students' thinking and understanding. Before being

> introduced to learning trajectories, I didn't realize how much variance there can be in understanding geometry (and all areas in math). I guess I knew it was there. I just didn't know there was a tool to guide me in determining where students are and where they need to go next. (Journal 5)

Melissa felt that LTs should be used "not just in math, but in every subject" (Interview 3) Ultimately, she saw LTs as a "philosophy of teaching," which was a rather profound statement.

And, lastly, Nan discussed LTs in conjunction with equity in education. In Interview 3, she said:

> Equal learning opportunities would be me as the teacher giving all students the same lesson. That's equal, right? But that is not what's going to help all of the students grow the same amount. If I want all of them to grow the same amount, so an equal amount of growing, it means that I interact with them differently, making it equitable rather than equal. That's where the learning trajectories become super-important, because this is ensuring that this is an equitable lesson, not an equal lesson.

Learning Trajectories have the potential to bridge many gaps. First of all, they can bridge the gap many teachers have in their own understanding of how different mathematical concepts progress. A teacher may know the mathematical content they need to teach at a given grade level. However, if they do not know where the concept came from and where it is going, it is highly unlikely that they can differentiate their instruction well enough to fit the needs of their students. Secondly, knowing the trajectories allows teachers to bridge the gap between what the student can do and where they want them to be. This applies to all students at all levels. LTs are not just a tool to differentiate instruction for struggling students. They are applicable to all students. Finally, as stated by the PSTs in this study, LTs help to create a cognitive road map of student's thinking.

This is a map that every teacher could use because it has the potential to show them where their students are (assessment), where their students are going (planning), and how they can get them there (instruction). And with this map, both the teacher and/or the students can take a different path at any time and still find their way.

Connecting Findings to Existing Literature

Over the past decade, the existing literature has analyzed the effects LTs have on teaching and learning (Bardsley, 2006; Bargagliotti & Anderson, 2017; Clements et al., 2011; Edgington, 2012; McCool, 2009; Mojica, 2010; Wickstrom & Langrall, 2018; Wilson et al., 2015), with positive results. These studies offer evidence that LTs aid teachers in focusing instruction on children's mathematical thinking. This study found evidence that PSTs: showed flexibility in identifying their students' mathematical thinking, planned lessons within their student's *zone of proximal development*, created lessons that encouraged active learning, and used LTs as a formative assessment tool through active learning. This study adds to the premise that LTs aid teachers in focusing instruction on children's mathematical thinking.

In the literature, McCool's (2009), used professional development approach with a teacher tutoring two students, which focused on LTs for a fifth-grade measurement topic to guide instructional decisions. The study reported that the LTs enabled the teacher to focus on the students' mathematical thinking and to use that information to make instructional decisions. The present study also investigated the impact LTs had on tutoring instruction of fifth-grade students, and added to the existing literature by examining preservice teachers, which found evidence that aligned with McCool's (2009) study.

Mojica (2010), also studied ways LTs were used by preservice teachers to build models of their students' mathematical thinking. The findings suggested not only that the PSTs created accurate models of the students' thinking and incorporated those models into their instruction, but LTs also increased the sophistication of their mathematics knowledge for teaching. And while the PSTs in the present study did not create models of their students thinking, they did reflect on LTs as a tool for creating a *cognitive road map* of students' thinking, which in it of itself is a version of a model of their students' thinking. In addition, this road map would be integral to instructional decisions.

The present study found evidence that PSTs used LTs as a tool for formative assessment. These results conflict, partially, with a recent study by Wickstrom and Langrall (2018), who followed a single teacher who participated in a 10 day professional development on an area measurement LT. The goal of the professional development was to prepare teachers to use the LT as a formative assessment tool in the classroom. They did find that the LTs supported the teacher in identifying and attending to student thinking, choosing appropriate lesson goals, and modifying tasks. However, the teacher chose to not use the LT to provide differentiated instruction, connect the lesson goals to the goals of the curriculum, or to formatively asses her students over time, which the present study did find.

Two major findings in the present study originated from data on PSTs' reflections on future use of LTs. PSTs considered LTs as a tool for grouping their students for instruction. In their reflection, small groups would consist of students with varying degrees of levels of proficiency, yet the ranges of the members of the group would not to be too wide. Their *hetero-homogeneous* small-groups would consist of knowledgeable others who are within the members' zone of proximal development. This aligns with existing literature. Clements and Sarama (2014) discuss collaborative learning and peer tutoring. In their example, teachers identify students needing help with specific topics and then matching them up with a peer who might teach that student. "These pairs and skills change frequently, so that all students have the opportunity to be ‘coaches’ and ‘players.” (p. 302).

During whole-group instructions, PSTs believed LTs could be used by the teacher to explore a topic and ask questions that increase along the trajectories, giving all students a chance to participate with contributions to, and learning from, the entire class. The present study adds to the understanding that effective instruction starts and ends with children’s mathematical thinking. This also aligns with existing literature. Clements and Sarama (2014) emphatically stress the importance of teachers talking about math with students and asking questions. “Expect children, as young as preschool, to share strategies, explain their thinking, work together to solve problems, and listen to each other” (p. 295). They also suggest that teachers, “Build on and elaborate children’s mathematical ideas and strategies” (p. 295).

Limitations of this Study

There were several limitations to this study. By selecting only preservice teachers in the capstone course, which is the final mathematics education course taken in the teacher education program, the researcher limited the possibility of understanding how preservice teachers in earlier courses might have been impacted by this study.

A limitation of this study was that the researcher selected the LT assessment tool, rather than providing students with options. The tool for this study had very specific goals and objectives in mind to help students increase their understanding of LTs. It is possible that other LT assessment tools may have had different results.

Another limitation of this study comes from the researcher’s relationship with the PSTs. The researcher taught all three PSTs in prior courses and had the opportunity to build rapport with them. This rapport may have influenced them to participate in the study. Similarly, it is also possible that once they agreed to participate, they may have been more serious about their work because of our built rapport and not wanting to disappoint the researcher.

Finally, the small number of three PSTs limits this study's findings from being generalizable. Twelve-weeks is a relatively short amount of time, further limiting this study. In addition, by working with only fifth grade students, it is possible that the PSTs' experience might be quite different had they worked with younger children. And, by only working with students one-on-one, this study limited the PSTs' from experiencing the use of LTs with small-group and whole-group settings, which could have had a profound impact on them, as well.

Implications for Teaching and Learning

Regarding the impact of this study, it was found that LTs aided PSTs in focusing on children's mathematical thinking during assessment, planning, and instruction (API). The increased attention to children's mathematical thinking paved the way for PSTs to become more flexible in all phases of API. Therefore, teacher education programs are encouraged to use LTs with PSTs in mathematics education methods courses as a tool to increase their knowledge for teaching. This could be done through one-on-one tutoring, similar to this study. In addition to being used as a tool with fieldwork, simply studying LTs in class and comparing them to the *Common Core State Standards in Mathematics* might have a positive impact on PSTs.

Additionally, the findings suggest that LTs encouraged PSTs to consider ways to differentiate their instruction, imagining them as a tool to create and use cognitive road maps of their student's thinking. Therefore, teacher education programs are encouraged to use LTs in conjunction with the topic of differentiation, specifically, when the strategy is focused on how to differentiate content during instruction.

The findings suggest that LTs have an important role to play in guiding PSTs with establishing small-groups for instruction. It is recommended that teacher educators use LTs with PSTs when exploring the importance of collaborative learning and peer tutoring, making sure they are aware of the significance of rearranging groups so that their future students play the role of teacher and learner, equally amongst their classmates.

Finally, the findings of this study support the use of LTs with PSTs when it comes to increasing their understanding of how to run effective whole-group discussions. All students can benefit from whole-group discussions in which the teachers ask questions in which students must explain their thinking and to listen to one another. Teacher educators are encouraged to use LTs in conjunction with training on how to ask probing questions that elicit students' thinking.

Recommendations for Future Research

The present study opens many doors to further research. First of all, given the small number of PSTs (3) in the present study, and the relatively short amount of time (12 weeks) spent with them, it is important to replicate the study on a larger scale. It would also be helpful for the researchers to not have prior relationships with the PSTs. If the findings can be replicated, it would be beneficial to study the impact on student learning during one-on-one tutoring. Next, studying PSTs at all phases of their teacher education program might help teacher educators understand if there are restrictions to when LTs should be introduced to PSTs. Similarly, having PSTs work with students in earlier grades, including pre-k, could be very beneficial. Finally, further research with PSTs working with students in small-group and whole-group settings is suggested.

It is recommended that other research-based LT tools besides Battista's (2012) be used with PSTs, to compare and contrast the results. It would be useful to do a follow-up study to find out the ways in which PSTs use LTs in their first year of teaching. It would also be interesting to know whether the results of this study translate beyond their first year through a longitudinal study.

McCool (2009) and Bargagliotti and Anderson (2017) found that LTs increased both the teachers' content and pedagogical knowledge for teaching. It would be relevant to know if the use of LTs increase PSTs' mathematical knowledge for teaching geometric shapes. Moreover, it would be interesting to know if LTs increase their van Hiele levels of geometric thinking.

Conclusion

In conclusion, the present study looked at how preservice teachers used learning trajectories during a one-on-one tutoring project. Additionally, the PSTs were asked to reflect on their experience and to consider how they might use LTs in their future teaching. The study found that during the assessment phase, PSTs opined that LTs gave them flexibility in identifying their students' level of mathematical thinking. While planning, PSTs were able to create lesson plans that were within their students' zone of proximal development and also encouraged active learning. And during instruction, LTs were used as a tool for formative assessment. When asked to reflect on how LTs might be used during small-group and whole-group instruction, PSTs surmised that LTs could be used to create hetero-homogeneous groupings and used to ask questions that increase in sophistication, respectively.

APPENDIX A

Behavioral Research Informed Consent
Title of Study: *Pre-service teachers and Learning Trajectories*

Principal Investigator (PI): Jeramy Lee Donovan
Mathematics Education
248-229-9067

Purpose

You are being asked to be in a research study of pre-service teachers' use of learning trajectories during instruction because you are currently in a teacher education program seeking elementary certification. This study is being conducted at Wayne State University and Neithercut Elementary School (pseudonym). The estimated number of study participants to be enrolled at Wayne State University and Neithercut Elementary School is about 3.
Please read this form and ask any questions you may have before agreeing to be in the study.

In this research study, factors that influence your mathematical instruction with 5^{th} grade students will be explored through direct observations and interviews.

Study Procedures

If you agree to take part in this research study, you will be asked to allow me to observe you planning, instructing, and assessing geometry lessons during a one-on-one tutoring project using tool to identify the level of understanding your student has about 4-sided shapes. First, you will take the van Hiele level test to obtain information on your current level of geometric understanding. Next, I will observe you working with a 5^{th} grade student two times. During the first one, you will be getting to know the child, followed up by you teaching that child three lessons. You will be required to create four lesson plans. You will also participate in 4 interviews, 3 of them individually and 1 with two of your peers who are also a part of the study. All observations and interviews will be audio-recorded, transcribed, and then deleted at that end of the study. Finally, you will be asked to complete 5 journal entries related to your experience throughout the study.

1. You will participate in a two-hour professional development session in which I work with you and the other participates by introducing you to the tools you will be using with the 5^{th} grader you will be working with. Next, you will do an hour long interview with a 5^{th} grader to discover their understanding of geometric shapes. You will be asked to do a pre and post journal entry about this experience. Afterwards, you will meet with me in a secure place to complete the first interview which will last for one hour. Next, you will teacher a lesson to the 5^{th} grade student which will last one hour. Again, you will do a pre and post journal about this experience, followed by a second interview with me. You will then complete a final journal entry describing your overall experience during the study.

This will be followed by a third, and final, one-on-one interview lasting one hour. Lastly, you will participate in a final interview in which you, along with the other two participants in the study, will share your experiences throughout the study. This final interview will last 90-minutes. Describe the chronological tasks the participants will do as part of the research study

2. You will work with a 5th grade student at Neithercut Elementary School four times. Each session will last about one hour. You will complete 5 journal entries that are approximately 2 pages in length. You will participate in 4 interview. Three of them will last one hour and be one-on-one, the fourth one will include the other two participants and last 90-minutes.
3. You will be asked questions about your thoughts about the tools you use during your time working with the 5th grade student. You have the option of not answering any question.
4. Your identity will remain confidential. You will be given a code name that will be used on journal entries, observation notes, and interview transcripts.

Benefits

As a participant in this research study, there will be no monetary benefit for you. You will, however, receive a copy of the CBA tool used during the study. You will also receive an increase in knowledge on teaching. Moreover, information from this study may benefit other people now or in the future.

Risks

There are no known risks at this time to participation in this study.

The following information must be released/reported to the appropriate authorities if at any time during the study there is concern that:

- child abuse has possibly occurred,

Study Costs

- Participation in this study will be of no cost to you.

Compensation
You will not be paid for taking part in this study.

Confidentiality

All information collected about you during the course of this study will be kept confidential to the extent permitted by law. You will be identified in the research records by a code name or number. Information that identifies you personally will not be released without your written permission. However, the study sponsor, the Institutional Review Board (IRB) at Wayne State University, or federal agencies with appropriate regulatory oversight [e.g., Food and Drug Administration (FDA), Office for Human Research Protections (OHRP), Office of Civil Rights (OCR), etc.) may review your records.

When the results of this research are published or discussed in conferences, no information will be included that would reveal your identity.

If audiotape recordings of you will be used for research or educational purposes, your identity will be protected or disguised. All audio-taped observations and interviews will be destroyed immediately after transcription. Your name will not be used and will be replaced by a code name. You will be able to review any transcriptions from the observations and interviews upon request.

Voluntary Participation/Withdrawal

Taking part in this study is voluntary. You have the right to choose not to take part in this study. If you decide to take part in the study you can later change your mind and withdraw from the study. You are free to only answer questions that you want to answer. You are free to withdraw from participation in this study at any time. Your decisions will not change any present or future relationship with Wayne State University or its affiliates, or other services you are entitled to receive.

The PI may stop your participation in this study without your consent. The PI will make the decision and let you know if it is not possible for you to continue. The decision that is made is to protect your health and safety, or because you did not follow the instructions to take part in the study

Questions

If you have any questions about this study now or in the future, you may contact Jeramy Donovan at the following phone number 248-229-9067. If you have questions or concerns about your rights as a research participant, the Chair of the Institutional Review Board can be contacted at (313) 577-1628. If you are unable to contact the research staff, or if you want to talk to someone other than the research staff, you may also call the Wayne State Research Subject Advocate at (313) 577-1628 to discuss problems, obtain information, or offer input.

Consent to Participate in a Research Study

To voluntarily agree to take part in this study, you must sign on the line below. If you choose to take part in this study you may withdraw at any time. You are not giving up any of your legal rights by signing this form. Your signature below indicates that you have read, or had read to you, this entire consent form, including the risks and benefits, and have had all of your questions answered. You will be given a copy of this consent form.

____ ____________
Signature of participant Date

____ ____________
Printed name of participant Time

____ ____________
Signature of person obtaining consent Date

____ ____________
Printed name of person obtaining consent Time

APPENDIX B

Timeline

Table 1. Timeline of study

Instructional Activities	Outcomes	Data
Week 1 January 30th, 2017		
• Pre-test administration (Van Hiele test)	• Identify potential PSTs for study	• Pre-test
Week 2 February 6th, 2017		
• Whole group	• Introduction to LTs and diagnostic tool	• Journal 1
Week 3 February 13th, 2017		
• Clinical interviews	• PSTs do a clinical interview with a 5th grader	• Clinical interview Results • Observations • Journal 2
Week 4 February 20th, 2017		
• Interview 1	• First individual interview (pre and post clinical interview)	• Interviews • Member check 1
Week 5 February 27th, 2017		
• Pre-lesson	• Prep for first lesson	• Journal 3
Week 6 March 6th, 2017		
• Lesson 1	• PSTs teach first lesson	• Observations • Lesson plans • Journal 4
Week 7 March 13th, 2017		
• Interview 2	• Second individual interview (pre and post lesson)	• Interviews • Member check 2
Week 8 March 20th, 2017		
• Lesson 2	• PSTs teach second lesson	• Observations • Lesson Plans
Week 9 March 27th, 2017		

• Lesson 3	• PSTs teach final lesson	• Observations • Lesson Plans
Week 10 April 6th, 2017		
• Final Lesson Plan	• Submission of final lesson plans	• Lesson Plans • Journal 5
a		
Week 11 April 17th, 2017		
• Interview 3	• Final individual interview (final thoughts)	• Interviews • Member check 3
Week 12 April 24th, 2017		
• Interview 4	• Focus group interview	• Interview • Member check 4

APPENDIX C

Van Hiele Test

Please print

Name ______________________________ Section _____________ .

Directions

Do not open this test booklet until you are told to do so.

This test contains 25 questions. It is not expected that you know everything on this test.

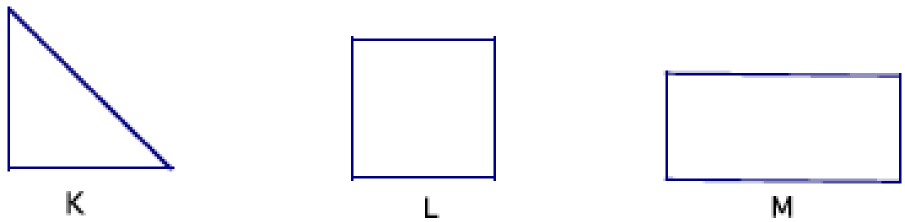
When you are told to begin:

1. Read each question carefully.
2. Decide upon the answer you think is correct. There is only one correct answer to each question. Check the letter corresponding to your answer on the answer sheet.
3. Use the space provided on the answer sheet for figuring or drawing. Do not mark on this test booklet.
4. If you want to change an answer, completely erase the first answer.
5. If you need a pencil and an eraser, raise your hand.
6. You will have 35 minutes for this test.

Wait until the instructor says that you may begin.

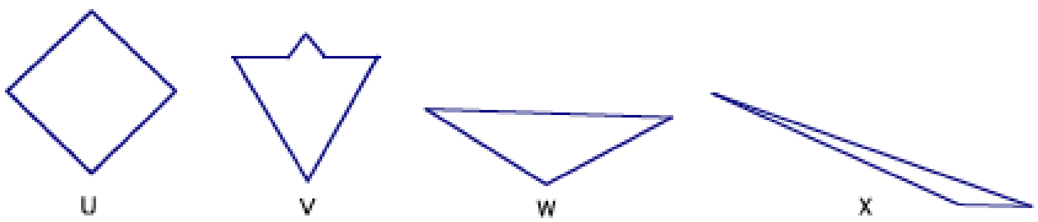
This test is based on the work of P.M. van Hiele.

1. Which of these are squares?



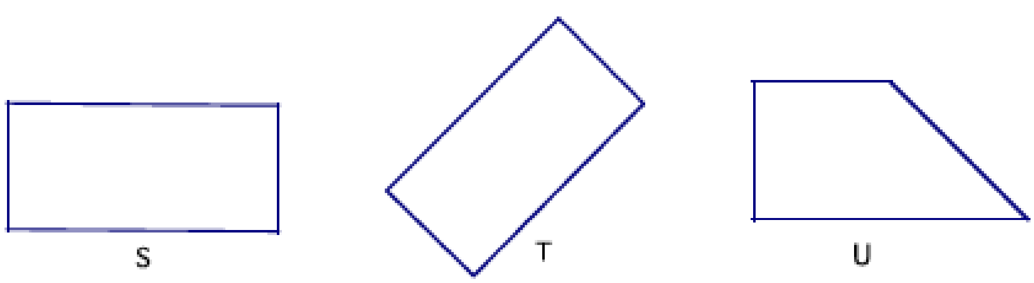
(F) K only
(G) L only
(H) M only
(I) L and M only
(J) All are squares

2. Which of these are triangles?



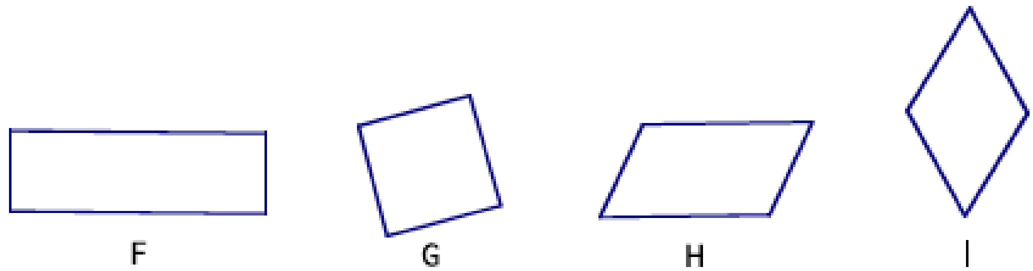
(A) None of these are triangles.
(B) V only
(C) W only
(D) W and X only
(E) V and W only

3. Which of these are rectangles?



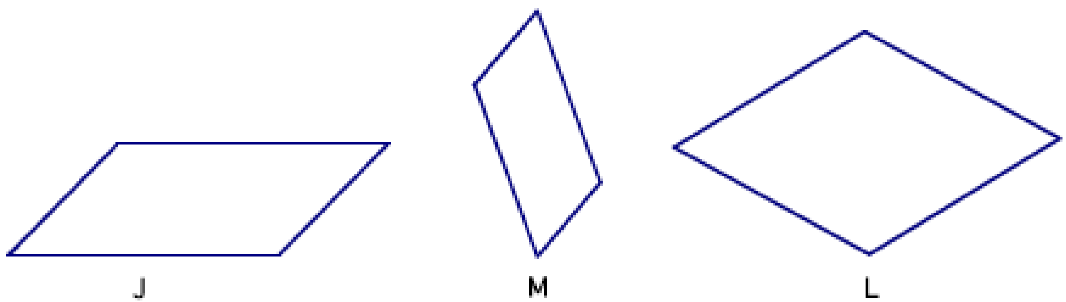
(A) S only
(B) T only
(C) S and T only
(D) S and U only
(E) All are rectangles

4. Which of these are squares?



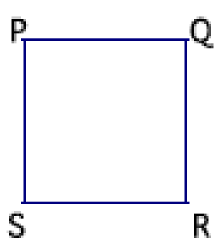
(A) None of these are squares.
(B) G only
(C) F and G only
(D) G and I only
(E) All are squares.

5. Which of these are parallelograms?



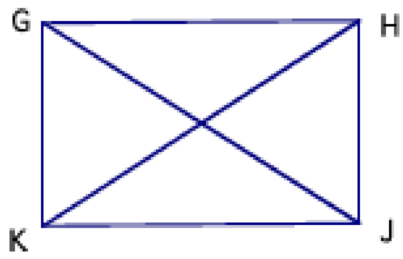
(A) J only
(B) L only
(C) J and M only
(D) None of these are parallelograms.
(E) All are parallelograms.

6. PQRS is a square.
Which relationship is true in all squares?



(A) $\overline{PR}$ and $\overline{RS}$ have the same length.
(B) $\overline{QS}$ and $\overline{PR}$ are perpendicular.
(C) $\overline{PS}$ and $\overline{QR}$ are perpendicular.
(D) $\overline{PS}$ and $\overline{QS}$ have the same length.
(E) Angle Q is larger than angle R.

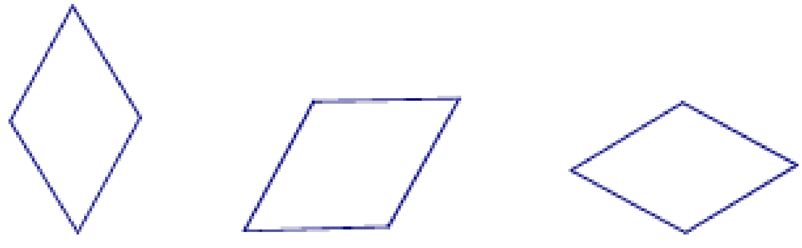
7. In a rectangle, GHJK, $\overline{GJ}$ and $\overline{HK}$ are the diagonals.



Which of (A) – (D) is not true in every rectangle?

(A) There are four right angles.
(B) There are four sides.
(C) The diagonals have the same length.
(D) The opposite sides have the same length.
(E) All of (A) – (D) are true in every rectangle.

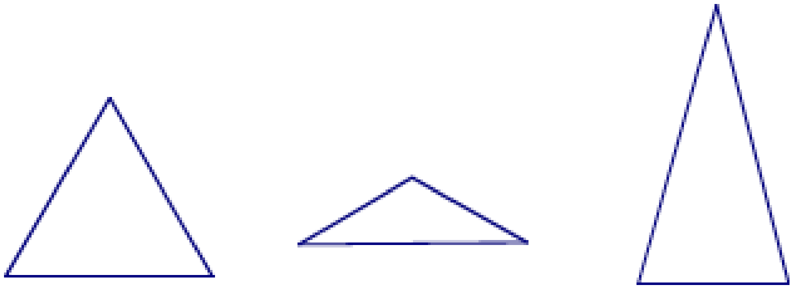
8. A rhombus is a 4-sided figure with all sides of the same length.
Here are three examples.



Which of (A) – (D) is not true in every rhombus?

(A) The two diagonals have the same length.
(B) Each diagonal bisects two angles of the rhombus.
(C) The two diagonals are perpendicular.
(D) The opposite angles have the same measure.
(E) All of (A) – (D) are true in every rhombus.

9. An isosceles triangle is a triangle with two sides of equal length.
Here are three examples.



Which of (A) – (D) is true in every isosceles triangle?

(A) The three sides must have the same length.
(B) One side must have twice the length of another side.
(C) There must be at least two angles with the same measure.
(D) The three angles must have the same measure.
(E) None of (A) – (D) is true in every isosceles triangle.

10. Two circles with centers P and Q intersect at R and S to form a 4-sided figure PRQS.
Here are two examples.

Which of (A) – (D) is <u>not</u> always true?

(A) PRQS will have two pairs of sides of equal length.
(B) PRQS will have at least two angles of equal measure.
(C) The lines $\overline{PQ}$ and $\overline{RS}$ will be perpendicular.
(D) Angles P and Q will have the same measure.
(E) All of (A) – (D) are true.

11. Here are two statements.

Statement 1: Figure F is a rectangle.
Statement 2: Figure F is a triangle.

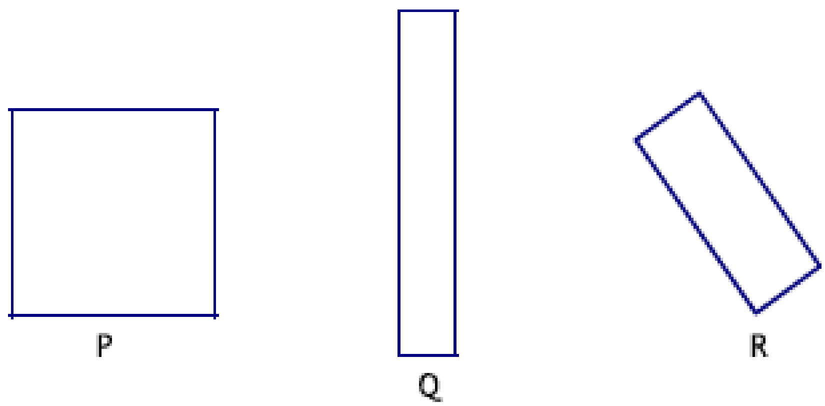
(A) If 1 is true, then 2 is true.
(B) If 1 is false, then 2 is true.
(C) 1 and 2 cannot both be true.
(D) 1 and 2 cannot both be false.
(E) None of (A) – (D) is correct.

12. Here are two statements.

Statement S: ΔABC has three sides of the same length.
Statement T: In ΔABC, $\angle B$ and $\angle C$ have the same measure.

Which is correct?
(A) Statements S and T cannot both be true.
(B) If S is true, then T is true.
(C) If T is true, then S is true.
(D) If S is false, then T is false.
(E) None of (A) – (D) is correct.

13. Which of these can be called rectangles?



(A) All can.
(B) Q only
(C) R only
(D) P and Q only
(E) Q and R only

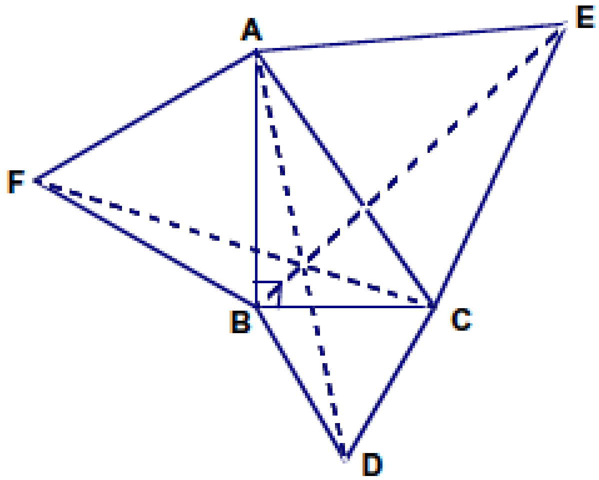
14. Which is true?

(A) All properties of rectangles are properties of all squares.
(B) All properties of squares are properties of all rectangles.
(C) All properties of rectangles are properties of all parallelograms.
(D) All properties of squares are properties of all parallelograms.
(E) None of (A) – (D) is true.

15. What do all rectangles have that some parallelograms do not have?

(A) Opposite sides equal
(B) Diagonals equal
(C) Opposite sides parallel
(D) Opposite angles equal
(E) None of (A) – (D)

16. Here is a right triangle ABC. Equilateral triangles ACE, ABF, and BCD have been constructed on the sides of ABC.



From this information, one can prove that $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ have a point in common. What would this proof tell you?

(A) Only this triangle drawn can we be sure that $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ have a point in common.
(B) In some but not all right triangles, $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ have a point in common.
(C) In any right triangle, $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ have a point in common.
(D) In any triangle, $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ have a point in common.
(E) In any equilateral triangle, $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ have a point in common.

17. Here are three properties of a figure.

    Property D: It has diagonals of equal length.
    Property S: It is a square.
    Property R: It is a rectangle.

    Which is true?
    (A) D implies S which implies R.
    (B) D implies R which implies S.
    (C) S implies R which implies D.
    (D) R implies D which implies S.
    (E) R implies S which implies D.

18. Here are two statements.

    I: If a figure is a rectangle, then its diagonals bisect each other.
    II: If the diagonals of a figure bisect each other, the figure is a rectangle.

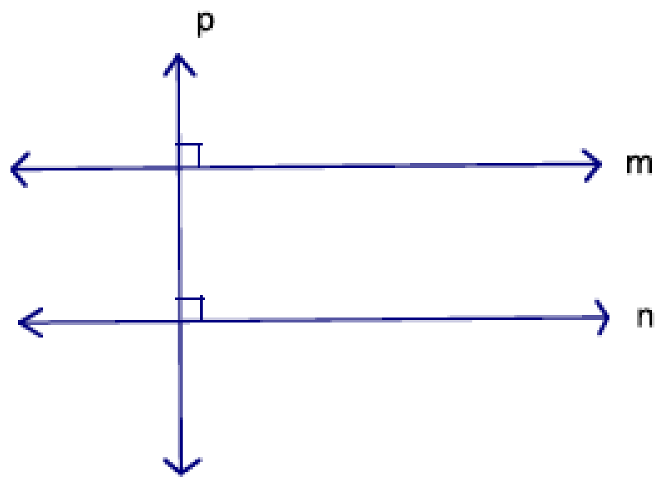
    Which is correct?
    (A) To prove I is true, it is enough to prove that II is true.
    (B) To prove II is true, it is enough to prove that I is true.
    (C) To prove II is true, it is enough to find one rectangle whose diagonal bisect each other.
    (D) To prove II is false, it is enough to find one non-rectangle whose diagonals bisect each other.
    (E) None of (A) – (D) is correct.

19. In geometry:

    (A) Every term can be defined and every true statement can be proved true.
    (B) Every term can be defined but it is necessary to assume that certain statements are true.
    (C) Some terms must be left undefined but every true statement can be proved true.
    (D) Some terms must be left undefined and it is necessary to have some statements, which are assumed true.
    (E) None of (A) – (D) is correct.

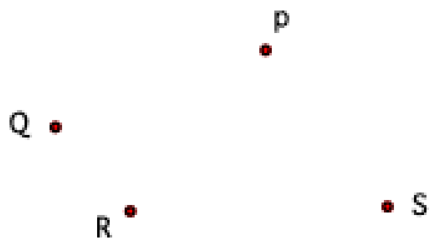
20. Examine these three sentences.
    (1) Two lines perpendicular to the same line are parallel.
    (2) A line that is perpendicular to one of two parallel lines is perpendicular to the other.
    (3) If two lines are equidistant, then they are parallel.

In the figure below, it is given that lines m and p are perpendicular and lines n and p are perpendicular. Which of the above sentences could be the reason that line m and is parallel to line n?



(A) (1) only
(B) (2) only
(C) (3) only
(D) Either (1) or (2)
(E) Either (2) or (3)

21. In F-geometry, one that is different from the one you are used to, there are exactly four points and six lines. Every line contains exactly two points. If the points are P, Q, R and S, the lines are {P, Q}, {P, R}, {P, S}, {Q, R}, {Q, S}, and {R, S}



Here are how the words "intersect" and "parallel" are used in F-geometry.
The lines {P, Q} and {P, R} intersect at P because {P, Q} and {P, R} have P in common.
The lines {P, Q} and {R, S} are parallel because they have no points in common.

From this information, which is correct?
(A) {P, R} and {Q, S} intersect.
(B) {P, R} and {Q, S} are parallel.
(C) {Q, R} and {R, S} are parallel.
(D) {P, S} and {Q, R} intersect.
(E) None of (A) – (D) is correct.

22. To trisect an angle means to divide it into three parts of equal measure. In 1847, P.L. Wantzal proved that, in general, it is impossible to trisect angles using only a compass and an unmarked ruler. From his proof, what can you conclude?

    (A) In general, it is impossible to bisect angles using only a compass and unmarked ruler.
    (B) In general, it is impossible to trisect angles using only a compass and marked ruler.
    (C) In general, it is impossible to trisect angles using any drawing instruments.
    (D) It is still possible that in the future someone may find a general way to trisect angles using only a compass and an unmarked ruler.
    (E) No one will ever be able to find a general method for trisecting angles using only a compass and an unmarked ruler.

23. There is a geometry invented by a mathematician J in which the following is true:

    The sum of the measures of the angles of a triangle is less than 180°.

    Which is correct?

    (A) J made a mistake in measuring the angles of the triangle.
    (B) J made a mistake in logical reasoning.
    (C) J has a wrong idea of what is meant by "true."
    (D) J started with different assumptions than those in the usual geometry.
    (E) None of (A) – (D) is correct.

24. Two geometry books define the word rectangle in different ways.
    Which is true?

    (A) One of the books has an error.
    (B) One of the definitions wrong. There cannot be two different definitions for rectangle.
    (C) The rectangles in one of the books must have different properties from those in the other book.
    (D) The rectangles in one of the books must have the same properties at those in the other book.
    (E) The properties of rectangles in the two books might be different.

25. Suppose you have proved statements I and II.

I. If p, then q.
II. If s, then not q.

Which statement follows from statements I and II?

(A) If p, then s.
(B) If not p, then not q.
(C) If p or q, then s.
(D) If s, then not p.
(E) If not s, then p.

APPENDIX D

Clinical Interview Questions

Appendix

CBA Assessment Tasks for Geometric Shapes

These problems can be used in individual interviews with children or in class as instructional activities. However, no matter which approach you choose, it is critical to get the students to describe and discuss their strategies. Only then can you use the CBA levels to interpret students' responses and decide on needed instruction.

Guide for Interviewing Students with CBA Tasks

The purpose of interviewing students with CBA tasks is to determine how they are reasoning and, more specifically, to determine what CBA levels of reasoning students are using for the tasks.

Before the interview, CBA teachers said the following to students:

> I am going to give you some problems. I would like to know what you think while you solve these problems. So, tell me everything you think as you do the problems. Try to think out loud. Tell me what you are doing and why you are doing it. I will also ask you questions to help me understand what you are thinking. For instance, if you say something that I don't understand, I will ask you questions about it.

If you don't understand what a student is saying, you could ask, "I don't understand, could you explain that again?" or "What do you mean by such-and-such?" Try to get students to explain in their own words, rather than paraphrasing what you think they mean and asking if they agree. If, during an interview, a student asks whether his or her answers are correct, we told the student that, for this interview, that does not really matter. We are interested in what he or she thinks.

Students responded to our request to "think out loud" in two ways. Many students were quite capable of thinking out loud as they solved problems. They told us what they were thinking and doing as they thought and did it. Other students, however, seemed unable to think aloud as they completed problems. They worked in silence, but then gave us detailed accounts of what they did *after* they finished doing it.

The following tasks cover a large range of geometric reasoning. You probably will not want to give all the problems to your students, at least not at one time. For students in grades 1–3, it is suggested that you select from Problems 1–5. For students in grades 4–8, it is suggested that you select from Problems 1–9. Of course, you can alter these suggestions based on your curriculum.

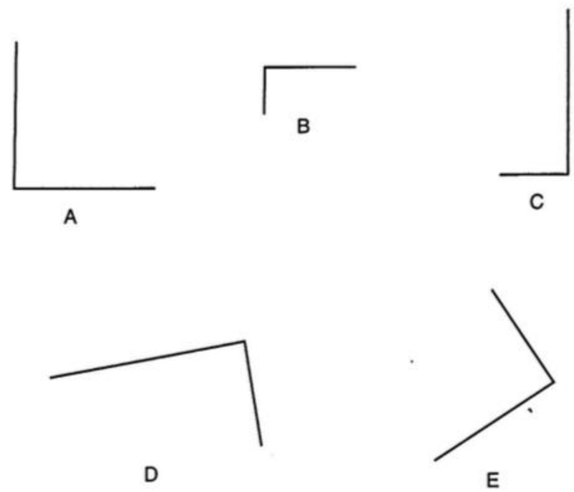
Problems 1–7 are especially helpful in deciding whether students' reasoning is in CBA Level 1 or 2. These problems can also help you decide which sublevel of reasoning students are using within Levels 1 and 2. Problems 8 and 9 are especially helpful for deciding sublevels within Level 3. If students ignore the measurements in Problems 6 and 7, there is no need to give them Problems 8 and 9.

Many of the problems have notes that indicate particular aspects of students' reasoning emphasized by the problems.

Additional assessment tasks may be downloaded from this book's website, www.heinemann.com/products/E04351.aspx. (Click on the "Companion Resources" tab.)

Name ______________________ Date ______________________

1. Drawing Machine 1 made these shapes. All shapes that can be made by Drawing Machine 1 are alike in some way. They follow a rule.



a. Describe how the shapes made by Drawing Machine 1 are alike. Tell what the rule is for this drawing machine.

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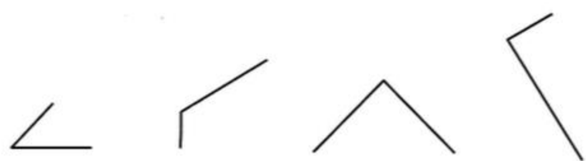
Name ______________________ Date ______________________

b. Drawing Machine 1 *cannot* make this shape. Why not?

F

c. Circle the shapes that Drawing Machine 1 *can* make. For each shape, tell why Drawing Machine 1 can or cannot make it.

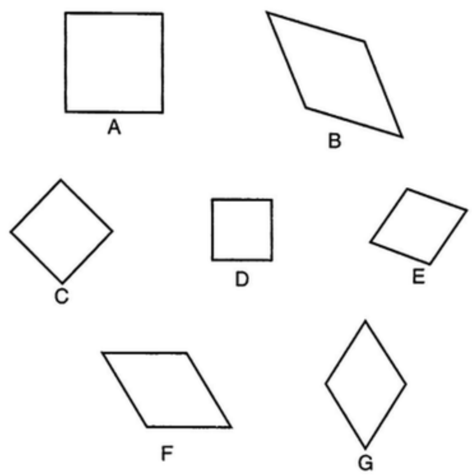


G H I J

Used with permission

Name ______________________________ Date ______________________________

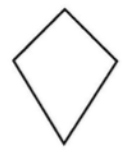
2. Drawing Machine 2 made these shapes. All shapes that can be made by Drawing Machine 2 are alike in some way. They follow a rule.



a. Describe how the shapes made by Drawing Machine 2 are alike. Tell what the rule is for this drawing machine.

Name ______________________________ Date ______________________________

b. Drawing Machine 2 *cannot* make this shape. Why not?



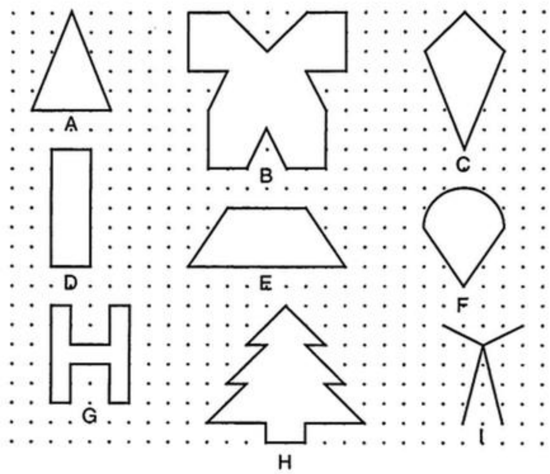
H

c. Circle the shapes that Drawing Machine 2 *can* make. For each shape, tell why Drawing Machine 2 can or cannot make it.

I J K L

Name ________________________________ Date ________________________________

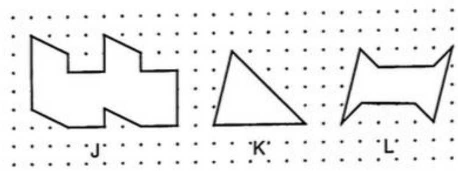
3. Drawing Machine 3 made these shapes. All shapes that can be made by Drawing Machine 3 are alike in some way. They follow a rule.



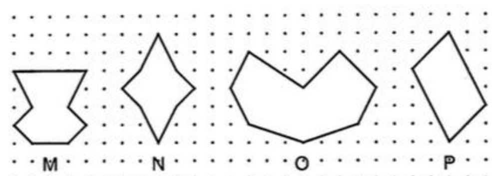
a. Describe how the shapes made by Drawing Machine 3 are alike. Tell what the rule is for this drawing machine.

Name ______________________ Date ______________________

b. Drawing Machine 3 *cannot* make these shapes. Why not?



c. Circle the shapes that Drawing Machine 3 *can* make. For each shape, tell why Drawing Machine 3 can or cannot make it.



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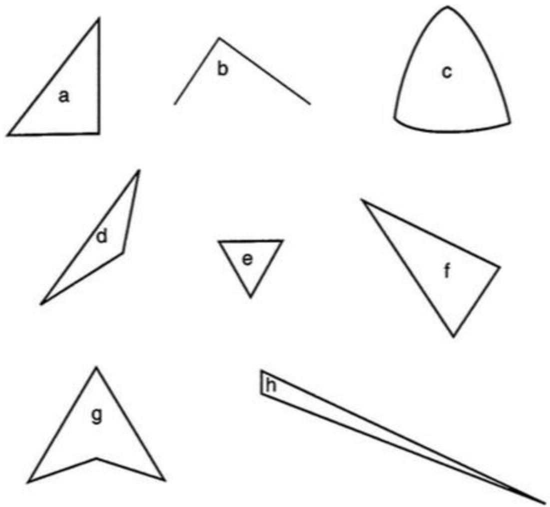
Name ______________________ Date ______________________

4. a. Circle each triangle.



b. Describe exactly how you decide if a shape is a triangle or not.

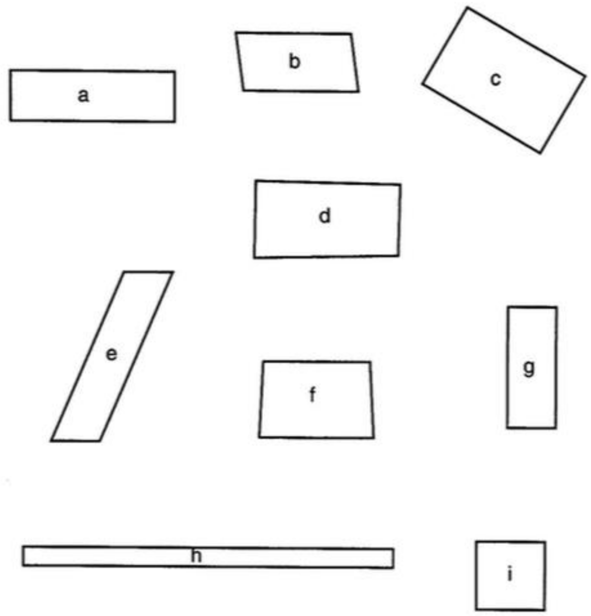
Name ______________________ Date ______________________

c. Is Shape *d* a triangle? Explain why.

d. Describe everything you know about triangles.

Name ______________________ Date ______________________

5. a. Circle each rectangle.



b. Describe exactly how you decide if a shape is a rectangle or not.

Name ______________________ Date ______________________

c. Is Shape *c* a rectangle? Explain why.

d. Describe everything you know about rectangles.

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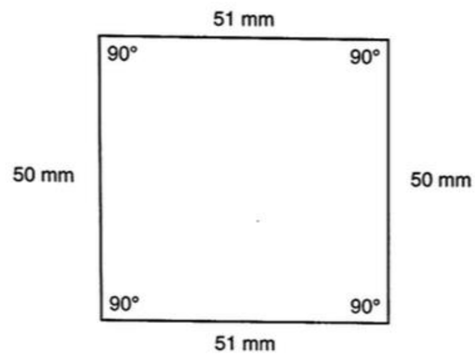
Name ______________________ Date ______________________

6. The measurements for a shape are given below.



For each statement about this shape, circle *T* if the statement is True, or *F* if the statement is False.
If you can't tell if the statement is true or false, circle *Can't tell*.
For each statement, describe how you would convince someone that your answer is correct.

a. The shape is a square. T F Can't tell

b. The shape is a rectangle. T F Can't tell

c. The shape is a parallelogram. T F Can't tell

d. The shape is a rhombus. T F Can't tell

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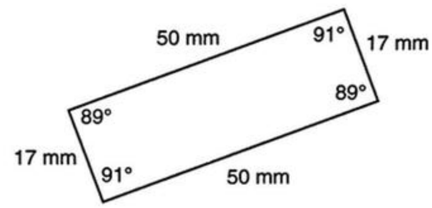
Name ______________________________ Date ______________________________

7. The measurements for a shape are given below.



For each statement about this shape, circle *T* if the statement is True, or *F* if the statement is False.
If you can't tell if the statement is true or false, circle *Can't tell*.
For each statement, describe how you would convince someone that your answer is correct.

a. The shape is a square. T F Can't tell

b. The shape is a rectangle. T F Can't tell

c. The shape is a parallelogram. T F Can't tell

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Name ______________________ Date ______________________

8. A quadrilateral is a closed shape with four straight sides. Examples of quadrilaterals are squares, rectangles, rhombuses, parallelograms, kites, and trapezoids.

Circle *True* if the statement is True, or *False* if the statement is False.
Describe how you would prove or show that your answer is correct.

If a quadrilateral has opposite sides equal and at least one right angle, then the quadrilateral is a rectangle.

Circle One: True False *Prove your answer or tell why your answer is correct.*

Name ______________________ Date ______________________

9. Tell whether the statement in the box is true or false.

Circle your answer. True False

All squares are rectangles.

What would you say to convince other students that your answer is correct?

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APPENDIX E

Journal Prompts

Prompt 1 - Pre-Clinical Interview

What do you plan to do in preparation for the interview?

What expectations do you have about yourself going into the clinical interview? Please include, but don't limit it to, your feelings of preparedness as an interviewer and your ability to draw out the information you need from your student in order to properly assess the child.

What are your expectations do you have about the student going into the clinical interview?

Prompt 2 - Post-Clinical Interview

Please share the results of the interview? Were you surprised? Explain.

What CBA level of sophistication is the child at? What evidence do you have?

What are the next steps for each child? How did you come to this conclusion?

Any other thoughts or comments?

Prompt 3 - Pre-Lesson

What is the purpose of this lesson and how does it relate to the larger goal of the student's reasoning about geometric shapes?

What hypotheses do you have in regards to the student's responses to the lesson activities?

How did you revise your lesson to meet the needs of your student? Support your reasoning?

Did the results of the CBA influence the lesson plan you used? Explain.

What role did the CBA play in your assessment planning?

What evidence will you be looking for in order to claim this as an effective lesson?

Any other thoughts or comments?

Prompt 4 - Post-Lesson

How effective do you think the lesson was in helping student increase his or her understanding of geometric shapes?

What particular areas of the lesson do you feel were most effective? Least?

What conclusions about their level of sophistication can you draw from the student's responses?

How did your predictions about the student play out?

Did the CBA results play a role in adapting your lesson on the spot? Explain

Describe what instructional moves supported the students' learning of geometric shapes and consider why tasks did or did not play out as intended.

What changes, if any, would you make to this lesson were you to teach it again?

What would you do with your student, next?

Any other thoughts or comments?

Prompt 5 - Final Thoughts

What impact has your experience with using LTs had on your thoughts about instruction (i.e., assessment, planning, and teaching)?

What role(s) do you believe LTs could play during individual, small-group, and a whole-class instruction? Describe how each scenario might look.

Most teachers have curriculums with scripted lessons and suggestions for student intervention. Do you believe LTs can still play a role in supporting instruction? Explain?

APPENDIX F

Interview Protocols

Development of the Interview Questions

Patton (2002) identified six types of questions that can be used during an interview: experience, opinion, feeling, knowledge, sensory, and background. Knowing and distinguishing these types of questions "forces the interviewer to be clear about what is being asked and helps the interviewer respond appropriately" (Patton, 2002, p. 348).

Whenever possible, each interview will follow Patton's (2002) sequencing of questions: describe or reconstruct; opinions and feelings; and knowledge and skills.

Elements of Siedman's (2006) recommendations on types of questions to ask, such as grand tour and mini tour questions, will be embedded in the interview questions, as well.

Lastly, advice from both Patton (2002) and Siedman (2006) on the wording of the questions, such as avoiding "why" questions, transitions, and probing, were all taken into consideration during the development of the interview questions.

Each of the four sets of interview protocols will be accompanied by the context of the setting and the research question(s) addressed.

Interview 1

Interview 1 partially addresses research question 1: In what ways do PSTs use LTs to assess, plan, and instruct lessons on a geometry topic? PSTs will only be doing an initial assessment prior to this interview; planning and instruction will be discussed the subsequent interviews.

Interview 1 sets the stage for looking into how PSTs begin to use LTs as a tool for assessing student knowledge of geometric shapes. Initially, a professional development session will introduce the PSTs to Learning Trajectories. Next, each participant will use Battista's (2012) book, *Cognition-Based Assessment & Teaching*, during a clinical interview to identify a 5^{th} grade student's level of sophistication of geometric shapes. Thereupon, they will determine "next steps" for instruction. Please note, no lesson plans or teaching will take place prior to this.

Interview 1 protocol

"Thanks for meeting with me today. I enjoyed observing your clinical interview with the 5th grade student. This interview should last approximately 45 minutes. I'll be audio-recording it along with taking notes throughout. The recording will be transcribed and I will be analyzing the results, looking for themes in your responses. Once I have done this, I will meet with you to go over my findings so that you can correct me on any misinterpretations. Do you have any questions before we begin?"

Grand Tour Question: - Interviewee is asked to reconstruct a significant segment of an experience.

1) Describe the clinical interview you did with the 5th grade student. *(possible probing questions to interpret the experience: What, if any, success did you have during the clinical interview?; Did you feel prepared to do the interview?; How did it make you feel?; What challenges did you face?)*

Mini Tour Question: - Interviewee is asked to reconstruct a more limited timeframe.

2) What was the most important moment during the clinical interview? Please take me through that moment. *(possible probing questions to interpret the experience: What, if any, success did you have during the clinical interview?; How did it make you feel?; What challenges did you face?)*

Transition: Do you have anything else to add or clarify before moving on?

Knowledge Questions:

3) What comes to mind when you think about "learning trajectories" in conjunction with "assessment"?

4) What role, if any, did learning trajectories play when considering the next steps to take with the student you interviewed?

Simulating Question: - Establishes neutrality between the interviewer and the interviewee

5) Suppose I was one of your classmates who has not been exposed to learning trajectories. What would you tell me about them?

Final Questions

6) What is your opinion about the use of learning trajectories to gain understanding about students' existing knowledge?

7) That covers the things I wanted to ask. Anything you care to add

Interview 2

Interview 2 addresses the remainder of research question 1: In what ways do PSTs use LTs to assess, plan, and instruct lessons on a geometry topic? Participants plan and instruct a lesson on geometric shapes.

Considering the "next steps" identified for their student, the PSTs will find a 5th grade lesson on geometric shapes and differentiate its content, if they feel inclined to do so, according to their student's level of sophistication. The PSTs will then teach the lesson and assess its effectiveness. Interview 2 continues with questions focusing on the PST's present lived experience, paying particular attention as to ask for "concrete details…before exploring attitudes and opinions about it" (Seidman, 2006, p. 88). In addition, PSTs will be asked to consider ways they used LTs during planning and instructing.

Interview 2 Protocol

Grand Tour Question

1) Take me through your experience with the lesson I observed. What happened? *(probing questions will be similar to the ones used in Interview 1)*

Mini-Tour Question

2) What was the most important moment in the lesson for you? Please take me through that moment *(possible probing questions to interpret the experience: What, if any, success did you have during it?; Did you feel prepared to do the interview?; How did it make you feel?; What challenges did you face?)*

Transition: Do you have anything else to add or clarify before moving on?

Knowledge Questions *(Please note that the next five questions are meant to establish "concrete details", as mentioned above)*

3) What does your student know about quadrilaterals. How do you know?

4) What role, if any, did LTs play during your lesson planning?

5) What role, if any, did LTs play during the lesson?

6) What comes to mind when you think about “learning trajectories” in conjunction with “lesson planning”?

7) What comes to mind when you think about “learning trajectories” in conjunction with “instruction”?

Final Question

8) That covers the things I wanted to ask. Anything you care to add?

Interview 3

Interview 3 addresses both research questions:

1) In what ways do PSTs use LTs to assess, plan, and instruct lessons on a geometry topic?
2) In what ways do PSTs reflect on their use of LTs and plan to use LTs to guide their future instruction?

Interview 3 takes place after the PSTs have developed a final lesson based on their student’s level of sophistication of geometric shapes. Also, PST have written their final thoughts into their journal, giving them time to reflect on the meaning of their experience throughout the study. This is the last one-on-one interview with the PSTs.

Interview 3 Protocol

Grand Tour Question:

1) During this study, you have been exposed to a number of things. Describe an experience you would like to share. *(PSTs are asked to reconstruct a significant experience, but one of their own choosing. The interviewee should not be expected to recreate the entire experience. Careful consideration will be taken to not interject so that the participant can discuss what they feel is most important to them)*

   Transition: Do you have anything else to add or clarify before moving on?

Knowledge Question

2) When you think about “learning trajectories” in conjunction with “teaching”, what comes to mind? *(The word “teaching” is used to imply planning, instruction, and assessment. Clarification will be made if needed, but the intent is to find out which one(s) they attend to)*

3) What role, if any, did LTs have throughout your instruction (i.e., planning, instructing, and assessing)?

Future Oriented Questions: - Patton (2002), suggest that these future oriented questions be saved for last, after present experiences have been explored, so that a baseline can be formed for them to build off of.

4) Given what you have experienced, how might you use LTs for future instruction?

5) How might a teacher use LTs in conjunction with their curriculum?

6) How might a teacher use LTs to think about whole group, small group, and individual instruction?

Future-Oriented/Role Playing Question: - Siedman (2006) likes these types of questions because they help to bring out an "inner voice" as opposed to a public voice. It also gives the interviewee a chance to play the role of "expert".

7) Pretend that you are in your first year of teaching. The teacher in the next room, whom you teach the same grade as, comes barging into your room begging for advice. He is having trouble teaching math. He says the struggling students are "too far behind" and the advanced students are "bored out of their minds". He doesn't know what to do. He tells you someone told him you have done work with learning trajectories. He wants to know if you think they can help him and, if so, how might he use them. Now, pretend I (the interviewer) am that teacher in need of help. What would your advice be? What would you tell me?

Final Question

8) That covers the things I wanted to ask. Anything you care to add?

Interview 4 – Focus Group Interview

Interview 4 takes place after I have had the chance to analyze all data collected, allowing me to look for themes to bring up during the focus group interview. I will address the two research questions for this study, but will also allow freedom for the group to discuss any topics that arise.

Many of the questions are similar to the Interview 3 questions. This was done on purpose. The focus group interview method assumes that individuals' attitudes and beliefs do not "form in a vacuum" and that listening to others' ideas benefit all involved (Marshall & Rossman, 2006, p. 114), which fits with the purpose of a collective case study, rather than focusing only on individual cases. This will allow the researcher to triangulate data with final written reflections and interviews.

The open-ended format of the focus group interview allows participants to comment, explain, and share experiences freely without a need for consensus (Krueger & Casey, 2000). It will be a nonjudgmental setting in which the experiences shared will allow a space for self-disclosure that might not happen in traditional, one-on-one interviews (Krueger & Casey, 2000).

Interview 4 Protocol

Grand Tour Question

1) Reconstruct something important you experienced throughout this study. What happened? *(While it will be expected that each participant shares an experience of their own choosing, participants will be encouraged to compare and contrast any experiences that come to mind while others are sharing, interjecting when appropriate)*

Knowledge Question

2) When I say "learning trajectories", what comes to mind? *(PSTs are asked only about LTs this time because I want them to make connections, if they so choose, to whatever they would like to.*

3) In what ways did LTs help you to think about instruction (i.e., planning, instructing, and assessing)?

Future Oriented Question

4) In what ways do you plan to use LTs in your future classroom?

Final Question

5) That covers the things I wanted to ask. Anything you care to add?

APPENDIX G

Blank Transcript Form

Date

PROJECT

TRANSCRIPT FORM

INTERVIEWER						
PARTICIPANT(s)						
DATE OF INTERVIEW						
INSTITUTION						
TYPE OF TRANSCRIPT	Interview		Focus Group		Other	

TRANSCRIPT	CODING & ANALYTIC MEMOS

APPENDIX H

Example of Transcript Form with Themes and Codes

Speaker	Transcript	Themes and Codes	Comments
Interviewee:	(11) It made me feel more confident when I was doing the interview, but I ... Even if I had needed to use them, I probably wouldn't have actually looked at them. The fact that I had written them out and searched for them was enough to have them in the forefront of of my mind.	LT Probing Questions - (11) Build Confidence	**Jeramy Donovan** Theme? – Creating interview probing questions – code would be that they build confidence.
Interviewer:	Got you. Yeah, that's amazing. Readily available for you. Good. In journal number two, which leads us into this, you mentioned that you're having difficulty placing your student at one specific level. You likened it to this idea of putting someone in a box. Remember that?		**Jeramy Donovan** Similar to Melissa's labeling issue
Interviewee:	(12) Mm-hmm (affirmative).	Challenge: - (12) Labeling/Classifying Student	
Interviewer:	When they never truly ... No one ever truly fits into that when ... In your opinion, do you feel it's necessary to choose one and only one level for a student when you're making these decisions?		
Interviewee:	(13) No. I just tried to generalize it and then I can always look ahead for different things to see ... (14) Maybe I can work at a higher level on this part, but I'll stay in the level I placed him for another part.	LT Flexibility - (13) Generalized Level labeling - (14) Can work with students on multiple levels	**Jeramy Donovan** Future Use Example
Interviewer:	You're comfortable working between levels with different ideas?		**Jeramy Donovan** LTs might not be causing a narrow vision/focus

APPENDIX I

Color-Coded Themes and Codes for a Single Case

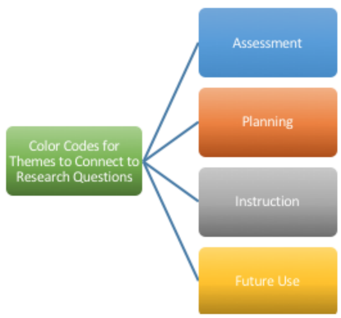
Themes & Codes – Interview 1 - B

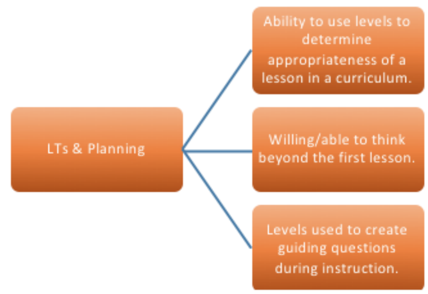
Themes & Codes – Journal 3 – B

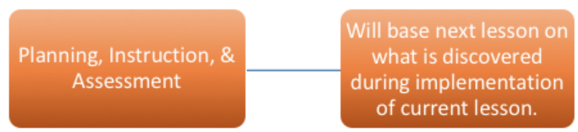
POST INTERVIEW 1

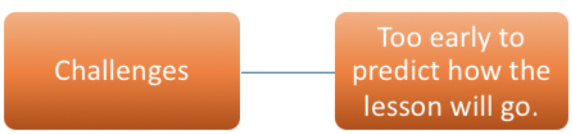
PRE LESSON 1

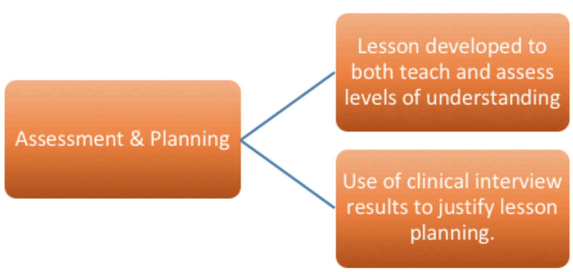
FOCUS ON PLANNING





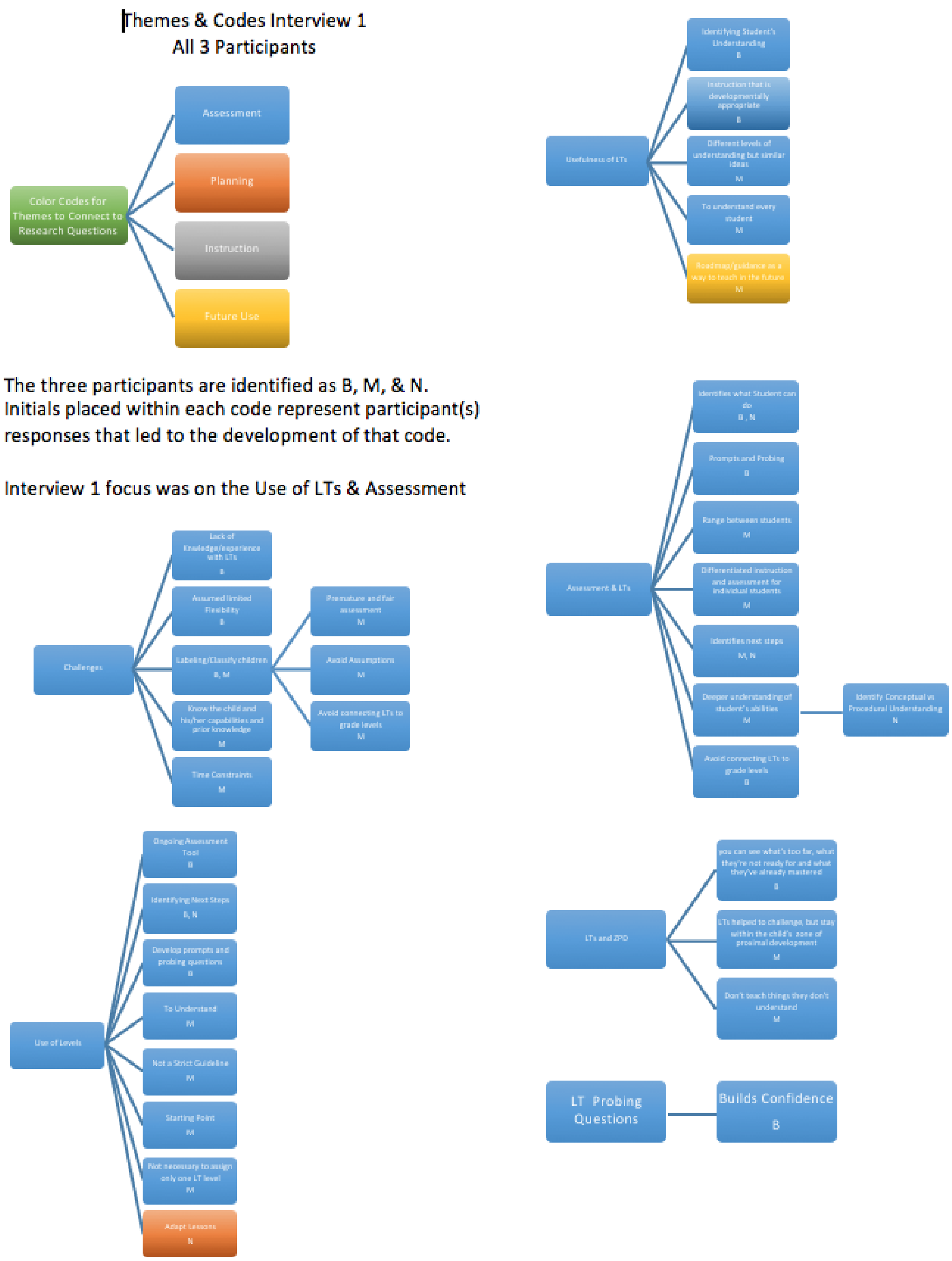


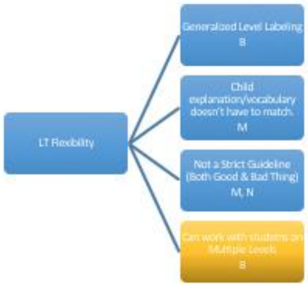




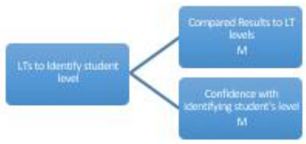
APPENDIX J

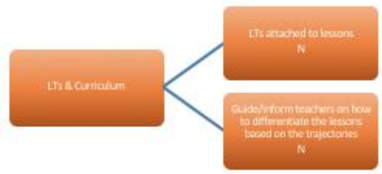
Color-Coded Themes and Codes for Collective Cases

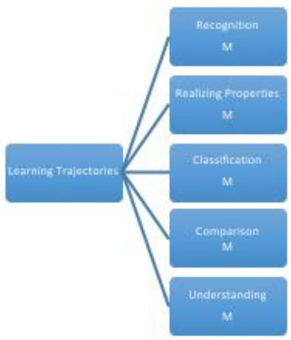


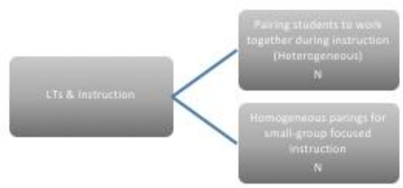

LT Flexibility
Generalized Level Labeling
B
Child explanation/vocabulary doesn't have to match.
M
Not a Strict Guideline (Both Good & Bad Thing)
M, N
Can work with students on Multiple Levels
B

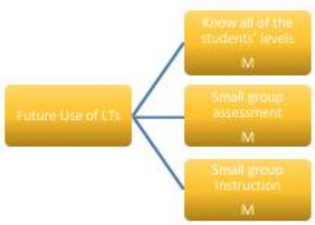

LTs & Planning
LTs don't add to the time needed to plan. Instead, they aid in allocating the time needed.
N
Identify class as a whole when planning
N


LTs to Identify student level
Compared Results to LT levels
M
Confidence with identifying student's level
M


LTs & Curriculum
LTs attached to lessons
N
Guide/inform teachers on how to differentiate the lessons based on the trajectories
N


Learning Trajectories
Recognition
M
Realizing Properties
M
Classification
M
Comparison
M
Understanding
M


LTs & Instruction
Pairing students to work together during instruction (Heterogeneous)
N
Homogeneous parings for small-group focused instruction
N


Future Use of LTs
Know all of the students' levels
M
Small group assessment
M
Small group instruction
M

I


Planning & Assessment
Must know where they are to know where to take them
N

APPENDIX K

Levels of Sophistication in Student Reasoning:

Geometric Shapes

Level 1.1 Student incorrectly identifies shapes as visual wholes.

Students' inaccurate visual-holistic reasoning prevents them from identifying many common shapes. One reason for this inaccuracy is that students at this level do not construct and operate on accurate images of shapes—their imagery distorts shapes in critically important ways.

Level 1.2 Student correctly identifies shapes as visual wholes.

Students use visual-holistic reasoning to *correctly* identify familiar geometric shapes. They often recognize when a group of shapes possesses a common set of geometric properties (although they cannot accurately describe these properties). Students at this level can form accurate visual images of geometric figures, and they can manipulate these images in ways that maintain the images' structure and identity.

Level 2.1 Student informally describes parts and properties of shapes.

Using language learned in everyday conversation, students describe parts and properties of shapes visually, informally, and imprecisely. For example, as informal references to the property of having all right angles, students might say that rectangles have "square corners" or "straight sides" or "sides that are not at an angle." When students say that rectangles and squares have "square corners," we have some evidence that they have noticed that rectangles and squares have the same special type of angles. But because the term *square corners* is not formally and precisely defined, it is difficult to know exactly what students mean. And when students say that rectangles have "straight sides" or "sides that are not at an angle," they are using the formal terms of *straight* and *angle* in informal ways, presumably as a reference to the perpendicularity of adjacent sides. However, sometimes when students use the term *straight* to describe a rectangle's angles, they mean that the sides of a rectangle are horizontal and vertical. (Think about when a picture hanging on a wall is considered "straight" or "not at an angle.")

Level 2.2 Student uses informal and insufficient formal descriptions of shapes.

When describing the parts or properties of shapes, students use a combination of informal and formal descriptions. The formal descriptions utilize standard geometric concepts and terms explicitly taught in mathematics curricula, but they are insufficient to completely specify shapes. For example, a student might explain that a rectangle has "sides across from each other that are equal [formal] and square corners [informal]." In this case, the formal portion of the student's description—that opposite sides are equal—is insufficient to specify rectangles. Students' formal language can be inadequate by being incorrect, insufficient, or inconsistent. Although students often recall properties they have abstracted for classes of shapes (say, "two long sides and two short sides" for rectangles), their reasoning is still visually based, and most of their descriptions and conceptualizations seem to occur extemporaneously as they are inspecting shapes.

Level 2.3 Student formally describes shapes completely and correctly.

Students use formal language and concepts to completely and correctly specify classes of shapes, and students' identification of shapes is consistent with their descriptions. Students have made a decided shift away from visually dominated reasoning because the major criterion for identifying a shape is whether it satisfies a set of verbally stated formal properties. So, for example, the term *rectangle* refers to a class of shapes that possesses all the properties the student has come to associate with the set of rectangles (e.g., opposite sides equal and parallel, and 4 right angles).

Students can use and create formal definitions for classes of shapes. However, the shape definitions that students create are not minimal; they list all the properties that they associate with the shape because they do not interrelate properties or recognize that some properties imply other properties.

Progressing to Level 2.3 requires that students understand formal concepts like side-length and angle-measure sufficiently so that they can use the concepts to describe important spatial relationships between shape parts.

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ABSTRACT

AN EXAMINATION OF PRESERVICE TEACHERS' USE OF LEARNING TRAJECTORIES TO GUIDE INSTRUCTION

by

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In order for teachers to support students' mathematical thinking, Battista (2004) believed they must identify core mathematical concepts, recognize conceptual frameworks for understanding children's thinking, and use appropriate assessment tasks. In his view, learning trajectories provide teachers with information on children's cognitive abilities as well as a structure for assessment. The present study investigated the ways in which preservice teachers used, and reflected on their use of, learning trajectories to assess, plan, and instruct during a one-on-one tutoring project focused on geometric shapes. In addition, preservice teachers were asked to reflect on the ways in which they might use learning trajectories during small-group and whole-group instruction.

This study employed collective case study methodology as a qualitative research design methodology. The goal of the research was to understand how three preservice teachers interpreted their experience. Participants were preservice teachers seeking K-8 teaching certification with a minor in mathematics education. Throughout the twelve-week study, digital recordings of fieldwork and interviews were collected, along with journal entries, lesson plans, and fieldnotes. The data analysis strategy followed Stake's (2006) methodology for collective case study analysis.

Trustworthiness was accomplished through thick description, triangulation of data, and member checks.

Supported by constructivist learning theory as the theoretical framework guiding the research, the study found that during the assessment phase, learning trajectories gave preservice teachers flexibility in identifying their students' level of mathematical thinking. While planning, preservice teachers created lesson plans that encouraged active learning and were within their students' zone of proximal development. And during instruction, learning trajectories were used as a tool for formative assessment. When asked to reflect on how learning trajectories might be used during small-group and whole-group instruction, preservice teachers surmised that learning trajectories could be used to create hetero-homogeneous groupings and used to ask questions that increase in sophistication, respectively.

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